ME 120 Experimental Methods

Uncertainty Analysis

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12SEP05

Goal of Uncertainty Analysis

- Come up with a probable bound on a measured value
  \[ x_{\text{true}} = x_{\text{meas}} \pm u_x \] at C\% confidence or n:1 odds
- The uncertainty, \( u_x \), will be calculated from the precision and bias errors, \( p_x \) and \( b_x \), associated with the measurement and measurement system.
  - The total uncertainty, \( u_x \) is:
    \[ u_x = (b_x^2 + p_x^2)^{1/2} \]
Classes of Experiments

- Two classes of experiments:
  - Single-sample (i.e., a single measurement of the measurand)
  - Repeat-sample (several measurements of the measurand under identical conditions)

(Note: “Sample” is used by some to refer to the set of data taken during repeated measurements of a measurand under fixed operating conditions. So a single-sample experiment would be a sample with only one measurement.)

Statistical Measurement Theory

- Precision errors are distributed, that is they follow a distribution that characterizes the probability that an error of a particular size will occur.
  - Ex. Gaussian or normal distribution
    - Most physical properties that are continuous or regular in time and space.
  - Other distributions are possible:
    - Log-normal (failure or durability projections)
    - Weibull (fatigue tests)
    - Poisson (events randomly occurring in time)
    - Binomial (reliability testing)

- The measurement sample (a limited set) is drawn from a population (the group of all measurements)
The Normal Distribution

- Precision error data is often *normally* distributed
  - The probability density function (PDF) for a normal distribution is:

\[
    f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
\]

- $x =$ value of measurement
- $\mu =$ mean of population
- $\sigma =$ standard deviation of population ($\sigma^2$ is called the variance)

(note that the mean and standard deviation of the population are usually unknown, so we will use estimates of them from a sample: $(\bar{x}, S_x)$)

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The Normal Distribution, cont.

- Standard Normal distribution:

\[
    z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}
\]

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Source: Beckwith, T. G., Marangoni, R. D., Lienhard, J. H., Mechanical Measurements, Addison-Wesley, Reading, MA, 1995
If the population statistics ($\mu$ and $\sigma$) are known, and assuming only precision errors, then with c% confidence, a single measurement is bounded by:

$$\mu - Z_{c/2} \sigma < x < \mu + Z_{c/2} \sigma$$

or alternatively,

we expect the measurand to lie in the range of:

$$\mu \pm Z_{c/2} \sigma \text{ with c% confidence}$$

where $Z_{c/2}$ is half the area under the standard normal distribution corresponding to c% probability.

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Ex. $\mu=200 \ ^\circ C$ $\sigma=3 \ ^\circ C$ for a particular chemical process from long term measurements. What range do we expect to see the temperature be in 95% of the time?

- Population or sample statistics?
- What type of distribution?
  - Plot the data on a histogram, and look at the shape
  - Plot the data on normal probability paper
  - Use a ‘goodness-of-fit’ test, like $\chi^2$

Assuming normally distributed data:

- Lookup $Z_{c/2}$, corresponding to 95% probability from the standard normal curve table
- Range will be: $200 \pm 1.96 \times (3 \ ^\circ C) = 200 \pm 5.88$ (should we round??)
Sample Statistical Parameters

- Population statistical parameters are usually unknown, so we need to rely on estimates

- **Sample mean** (for n measurements): \( \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \)

- **Sample standard deviation** (n measurements):
  \[
  S_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}} = \sqrt{\frac{(\sum_{i=1}^{n} x_i^2) - n\bar{x}^2}{n - 1}}
  \]

Sample Statistical Parameters, cont.

- Suppose m replicates of n samples
  - What do you expect to see with \( \bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_m \)?
  - The sample *means* have a distribution!
    - The distribution of sample means tends toward normal as n increases! (the Central Limit Theorem)
    - There will be a standard deviation associated with the distribution of sample means (also called the *estimated standard error*):
      \( S_{\bar{x}} = \frac{S_x}{\sqrt{n}} \)

What happens when the sample size increases?
Uncertainty Bounds from Sample Statistics

- We expect the true mean value of the measurand to lie in the range of:
  \[
  \bar{x}_{true} = \bar{x} \pm t_{\alpha/2,\nu}S_x
  \]
  (at C\% confidence)

  where
  - \(\bar{x}_{true}\) is the true mean
  - \(\bar{x}\) is the sample mean
  - \(t_{\alpha/2,\nu}\) is Student's t distribution with \(\nu\) degrees of freedom
  - \(\alpha\) is the level of significance (and \(C=1-\alpha\))

  \(\nu = n-1\)

  (note as \(n \to \infty\), \(t_{\alpha/2,\nu}\) approaches \(Z_{\alpha/2}\))

  Who was Student?

Example

<table>
<thead>
<tr>
<th>Sample No</th>
<th>Value</th>
<th>C% = 95%</th>
<th>(t_{\alpha/2}) = 0.025</th>
<th>Number of samples</th>
<th>Degrees of freedom</th>
<th>Sample average</th>
<th>Sample standard deviation</th>
<th>Estimated standard error</th>
<th>(t)-stat</th>
<th>(t)-stat (Sxbar)</th>
<th>95% Confidence</th>
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<td>Degrees of freedom</td>
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<td>(Sxbar = 0.035258)</td>
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Comparison of Sample Means

◆ The t distribution is also useful in determining whether or not the means of two populations are significantly different
  
  Calculate the the t-statistic: \[ t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}} \]

◆ See if it falls in the range of \( \pm t_{\alpha/2,\nu} \),

Where \( \nu = \left[ \frac{(S_1^2 / n_1) + (S_2^2 / n_2)}{\frac{(S_1^2 / n_1)^2}{n_1 - 1} - \frac{(S_2^2 / n_2)^2}{n_2 - 1}} \right] \) (rounded down)

If it does, than the population means are not significantly different at the stated level of confidence.

Confidence Interval on Standard Deviation

◆ The distribution describing the variation in standard deviations of samples is Chi-squared, so a bound on \( s \) can be calculated as:

\[
\sqrt{\frac{(N-1)s^2}{\chi^2_{(\alpha/2,N-1)}}} \leq \sigma \leq \sqrt{\frac{(N-1)s^2}{\chi^2_{(1-\alpha/2,N-1)}}}
\]

\( C\%=(1-\alpha) \)

Design-Stage Uncertainty Analysis

- What uncertainty arises before we start taking data?
  - Uncertainty from ability of measurement instrument to resolve
    - \( U_{res} = \pm \frac{1}{2} \) resolution (at C% Confidence)
  - Uncertainty from instrument errors
    - \( U_{ins} = \pm \sqrt{\sum_{i=1}^{k} e_i^2} \) (C% confidence)

Instrument Error Calculation

- Ex: LCKD “button” load cell from Omega
  - http://www.omega.com/Pressure/pdf/LCKD.pdf

  **SPECIFICATIONS:**
  - Excitation: 5 Vdc, 7 Vdc max
  - Output: 2 mV/V nominal
  - 5-Point Calibration: 0%, 50%, 100%, 50%, 0%
  - Linearity: \( \pm 0.25\% \) FSO
  - Hysteresis: \( \pm 0.25\% \) FSO
  - Repeatability: \( \pm 0.10\% \) FSO
  - Zero Balance: \( \pm 0.2\% \) FSO
  - Operating Temp Range: \(-54 \text{ to } 107^\circ \text{C}\) (\(-65 \text{ to } 225^\circ \text{F}\))
  - Compensated Temp Range: \(16 \text{ to } 71^\circ \text{C} \) (\(60 \text{ to } 160^\circ \text{F}\))
  - Thermal Effects:
    - Span: \( \pm 0.01\% \) of FSO/°F
    - Zero: \( \pm 0.05\% \) of FSO/°F
    - Safe Overload: 150% of Capacity
    - Ultimate Overload: 300% of Capacity
  - Weight: < 0.5 oz (< 14 g)

Suppose 0-100 lb FS range used at 80 °C, and that zero balance can be adjusted out:

\[
U_{ins} = \pm \sqrt{\left(\text{lin.}\right)^2 + \left(\text{hyst.}\right)^2 + \left(\text{repeat.}\right)^2 + \left(\text{z.h.}\right)^2 + \left(\text{th. span}\right)^2 + \left(\text{th. zer.}\right)^2}
\]

Note: this is the instrument (device) uncertainty, which is one part of the overall uncertainty for any particular measurement using this device. What else will contribute to the total uncertainty, and how will you calculate the total uncertainty?
Error Propagation

“When experimental data are used to compute a final result, the uncertainty of the data must be \textit{propagated} to determine the uncertainty in the result.” (BM&L, p. 116)

\[
U_{f(x_1, x_2, \ldots, x_n)} = \sqrt{\left( \frac{\partial f}{\partial x_1} u_1 \right)^2 + \left( \frac{\partial f}{\partial x_2} u_2 \right)^2 + \cdots + \left( \frac{\partial f}{\partial x_n} u_n \right)^2}
\]

\textbf{Ex:} \quad I = \frac{1}{12}wh^3 \quad \text{where w and h are measured values}

References

References, cont.

- Dallal, G. E., “What Student Did,”

Putting it all together