San Jose State University
Department of Mechanical and Aerospace Engineering

ME 130 Applied Engineering Analysis

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Chapter 8

Matrices and Solution to Simultaneous Equations by Gaussian Elimination Method

Chapter Outline

- Matrices and Linear Algebra
- Different Forms of Matrices
- Transposition of Matrices
- Matrix Algebra
- Matrix Inversion
- Solution of Simultaneous Equations
  - Using Inverse Matrices
  - Using Gaussian Elimination Method

Linear Algebra and Matrices

Linear algebra is a branch of mathematics concerned with the study of:

- Vectors
- Vector spaces (also called linear spaces)
- systems of linear equations

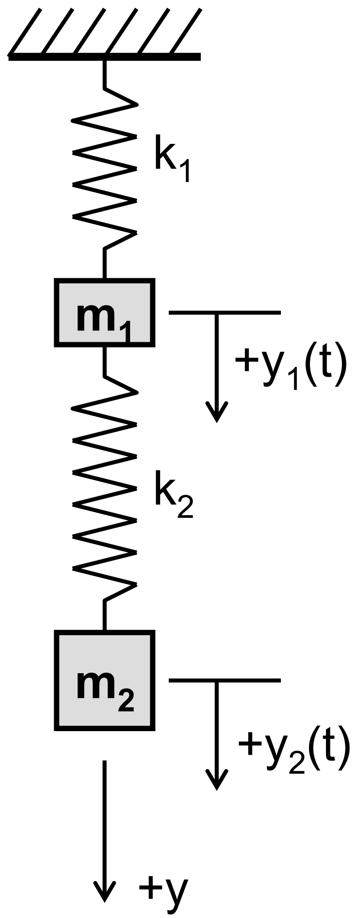
(Source: Wikipedia 2009)

Matrices are the logical and convenient representations of vectors in vector spaces, and

Matrix algebra is for arithmetic manipulations of matrices. It is a vital tool to solve systems of linear equations

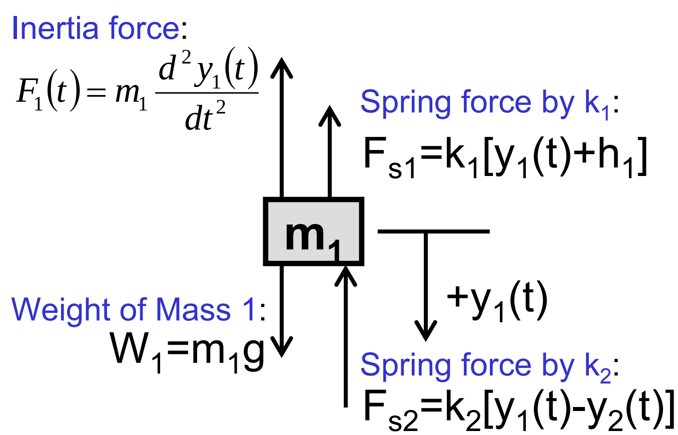
Systems of linear equations are common in engineering analysis:

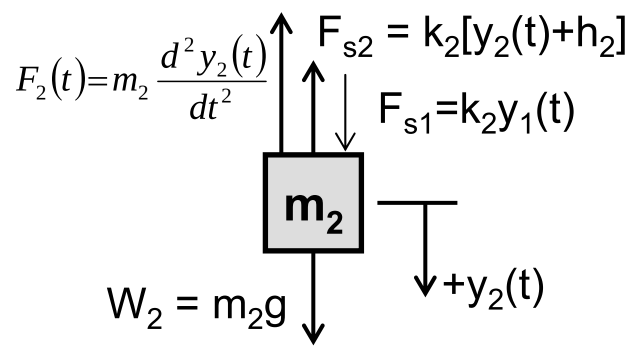
A simple example is the free vibration of Mass-spring with 2-degree-of freedom:



As we postulated in single mass-spring systems, the two masses m_1 and m_2 will vibrate prompted by a small disturbance applied to mass m_2 in the +y direction

Following the same procedures used in deriving the equation of motion of the masses using Newton's first and second laws, with the following free-body of forces acting on m_1 and m_2 at time t:





where W_1 and W_2 = weights; h_1, h_2 = static deflection of spring k_1 and k_2

A system of 2 simultaneous linear DEs for amplitudes $y_1(t)$ and $y_2(t)$:

$$m_1 \frac{d^2 y_1(t)}{dt^2} + (k_1 + k_2) y_1(t) - k_2 y_2(t) = 0$$

$$m_2 \frac{d^2 y_2(t)}{dt^2} + k_2 y_2(t) - k_2 y_1(t) = 0$$

Matrices

- Matrices are used to express **arrays** of numbers, variables or data in a logical format that can be accepted by digital computers
- Matrices are made up with ROWS and COLUMNS
- Matrices can represent vector quantities such as force vectors, stress vectors, velocity vectors, etc. All these vector quantities consist of several components
- Huge amount of numbers and data are common place in modern-day engineering analysis, especially in numerical analyses such as the finite element analysis (FEA) or finite deference analysis (FDA)

Different Forms of Matrices

1. **Rectangular matrices:**
   The total number of rows (m) ≠ The total number of columns (n)

Column (n): n=1, n=2, n=3, n = n

Abbreviation of Rectangular Matrices:

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix}$$

Matrix elements (a_{ij})

Rows (m): m =1, m =2, m = m

2. Square matrices:

It is a special case of rectangular matrices with:

The total number of rows (m) = total number of columns (n)

Example of a 3x3 square matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

diagonal line

- All square matrices have a "diagonal line"
- Square matrices are most common in computational engineering analyses

3. Row matrices:

Matrices with only one row, that is: m = 1:

$$\{A\} = \{a_{11} \quad a_{12} \quad a_{13} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad a_{1n}\}$$

4. Column matrices:

Opposite to row matrices, column matrices have only one column (n = 1), but with more than one row:

$$\{A\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ a_{m1} \end{Bmatrix}$$

column matrices represent vector quantities in engineering analyses, e.g.:

A force vector: $\{F\} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$

in which F_x, F_y and F_z are the three components along x-, y- and z-axis in a rectangular coordinate system respectively

5. Upper triangular matrices:

These matrices have all elements with "zero" value under the diagonal line

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

diagonal line

6. Lower triangular matrices

All elements above the diagonal line of the matrices are zero.

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

diagonal line

7. Diagonal matrices:

Matrices with all elements except those on the diagonal line are zero

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Diagonal line

8. Unity matrices [I]:

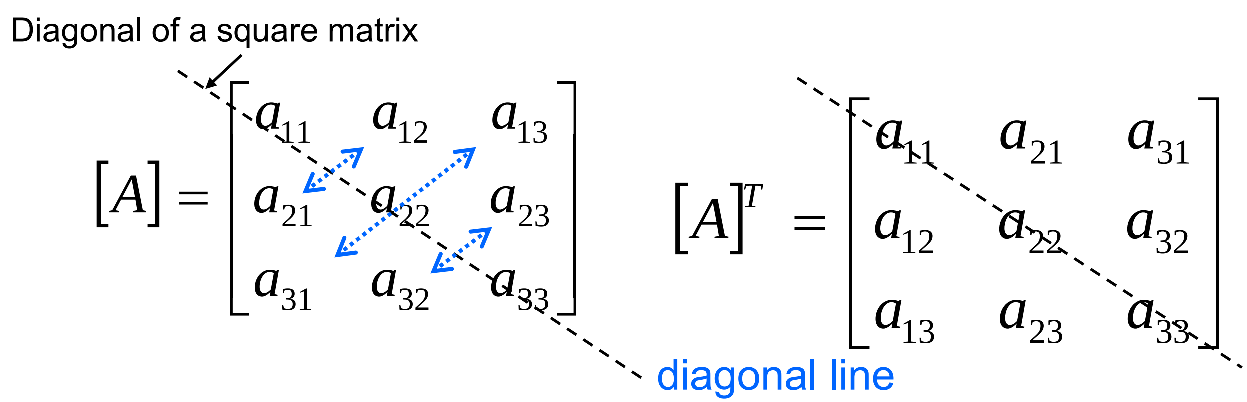
It is a special case of diagonal matrices, with elements of value 1 (unity value)

$$[I] = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

diagonal line

Transposition of Matrices:

- It is a common procedure in the manipulation of matrices
- The transposition of a matrix [A] is designated by $[A]^T$
- The transposition of matrix [A] is carried out by interchanging the elements in a square matrix across the diagonal line of that matrix:



(a) Original matrix (b) Transposed matrix

Matrix Algebra

- Matrices are expressions of ARRAY of numbers or variables. They CANNOT be deduced to a single value, as in the case of determinant
- Therefore matrices ≠ determinants
- Matrices can be summed, subtracted and multiplied but cannot be divided
- Results of the above algebraic operations of matrices are in the forms of matrices
- A matrix cannot be divided by another matrix, but the “sense” of division can be accomplished by the inverse matrix technique

1) Addition and subtraction of matrices

The involved matrices must have the SAME size (i.e., number of rows and columns):

$$[A] \pm [B] = [C] \ \ with \ \ elements \ c_{ij} = a_{ij} \pm b_{ij}$$

2) Multiplication with a scalar quantity (α):

$$\alpha [C] = [\alpha c_{ij}]$$

3) Multiplication of 2 matrices:

Multiplication of two matrices is possible only when:

The total number of columns in the 1st matrix
= the number of rows in the 2nd matrix:

$$\underset{(m \times p)}{[C]} = \underset{(m \times n)}{[A]} \times \underset{(n \times p)}{[B]}$$

The following recurrence relationship applies:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots\ldots a_{in}b_{nj}$$

with i = 1, 2, ….,m and j = 1,2,….., n

Example 8.1

Multiple the following two 3x3 matrices

$$[C] = [A][B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

So, we have:

$$\underset{(3x3)}{[A]} \quad x \quad \underset{(3x3)}{[B]} = \underset{(3x3)}{[C]}$$

Example 8.2:

Multiply a rectangular matrix by a column matrix:

$$[C]\{x\} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \{y\}$$

(2x3)x(3x1) (2x1)

Example 8.3:

(A) Multiplication of row and column matrices

$$\{a_{11} \quad a_{12} \quad a_{13}\} \begin{Bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{Bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \quad (a\ scalar\ or\ a\ single\ number)$$

(1x3) x (3x1) = (1x1)

(B) Multiplication of a column matrix and a square matrix:

$$\begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} \{b_{11} \quad b_{12} \quad b_{13}\} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{bmatrix} \quad (a\ square\ matrix)$$

(3x1) x (1x3) = (3x3)

Example 8.4
Multiplication of a square matrix and a row matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{Bmatrix} \qquad (a\ column\ matrix)$$

(3x3) x (3x1) = (3x1)

NOTE:

Because of the rule: **[C]** (m x p) **=** **[A]** (m x n) **x** **[B]** (n x p) we have $[A][B] \neq [B][A]$

Also, the following relationships will be useful:

Distributive law: [A]([B] + [C]) = [A][B] + [A][C]

Associative law: [A]([B][C]) = [A][B]([C])

Product of two transposed matrices: $([A][B])^T = [B]^T[A]^T$

Matrix Inversion

The inverse of a matrix [A], expressed as **$[A]^{-1}$**, is defined as:

$$[A][A]^{-1} = [A]^{-1}[A] = [I] \qquad (8.13)$$

(a UNIT matrix)

NOTE: the inverse of a matrix [A] exists ONLY if $|A| \neq 0$

where $|A|$ = the equivalent determinant of matrix [A]

Following are the general steps in inverting the matrix [A]:

Step 1**:** Evaluate the equivalent determinant of the matrix. Make sure that $|A| \neq 0$

Step 2**:** If the elements of matrix [A] are a_{ij}, we may determine the elements of a co-factor matrix [C] to be:

$$c_{ij} = (-1)^{i+j}\,|A'| \qquad (8.14)$$

in which $|A'|$ is the equivalent determinant of a matrix [A'] that has all elements of [A] excluding those in i^{th} row and j^{th} column.

Step 3**:** Transpose the co-factor matrix, [C] to $[C]^T$.

Step 4**:** The inverse matrix $[A]^{-1}$ for matrix [A] may be established by the following expression:

$$[A]^{-1} = \frac{1}{|A|}[C]^T \qquad (8.15)$$

Example 8.5

Show the inverse of a (3x3) matrix:

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{bmatrix}$$

Let us derive the inverse matrix of [A] by following the above steps:

Step 1: Evaluate the equivalent determinant of [A]:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ -2 & -3 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} = -39 \neq 0$$

So, we may proceed to determine the inverse of matrix [A]

Step 2: Use Equation (8.14) to find the elements of the co-factor matrix, [C] with its elements evaluated by the formula:

$$c_{ij} = (-1)^{i+j}\,|A'|$$

where $|A'|$ is the equivalent determinant of a matrix [A'] that has all elements of [A] excluding those in i^{th} row and j^{th} column.

$$c_{11} = (-1)^{1+1}[(-1)(-3)-(4)(5)] = -17$$
$$c_{12} = (-1)^{1+2}[(0)(-3)-(4)(-2)] = -8$$
$$c_{13} = (-1)^{1+3}[(0)(5)-(-1)(-2)] = -2$$
$$c_{21} = (-1)^{2+1}[(2)(-3)-(3)(5)] = 21$$
$$c_{22} = (-1)^{2+2}[(1)(-3)-(3)(-2)] = 3$$
$$c_{23} = (-1)^{2+3}[(1)(5)-(2)(-2)] = -9$$
$$c_{31} = (-1)^{3+1}[(2)(4)-(3)(-1)] = 11$$
$$c_{32} = (-1)^{3+2}[(1)(4)-(3)(0)] = -4$$
$$c_{33} = (-1)^{3+3}[(1)(-1)-(2)(0)] = -1$$

We thus have the co-factor matrix, [C] in the form:

$$[C] = \begin{bmatrix} -17 & -8 & -2 \\ 21 & 3 & -9 \\ 11 & -4 & -1 \end{bmatrix}$$

Step 3: Transpose the [C] matrix:

$$[C]^T = \begin{bmatrix} -17 & -8 & -2 \\ 21 & 3 & -9 \\ 11 & -4 & -1 \end{bmatrix}^T = \begin{bmatrix} -17 & 21 & 11 \\ -8 & 3 & -4 \\ -2 & -9 & -1 \end{bmatrix}$$

Step 4: The inverse of matrix [A] thus takes the form:

$$[A]^{-1} = \frac{[C]^T}{|A|} = \frac{1}{-39}\begin{bmatrix} -17 & 21 & 11 \\ -8 & 3 & -4 \\ -2 & -9 & -1 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 17 & -21 & -11 \\ 8 & -3 & 4 \\ 2 & 9 & 1 \end{bmatrix}$$

Check if the result is correct, i.e., $[A]\,[A]^{-1} = [\,I\,]$?

$$[A][A]^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{bmatrix}\left(\frac{1}{39}\right)\begin{bmatrix} 17 & -21 & -11 \\ 8 & -3 & 4 \\ 2 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I]$$

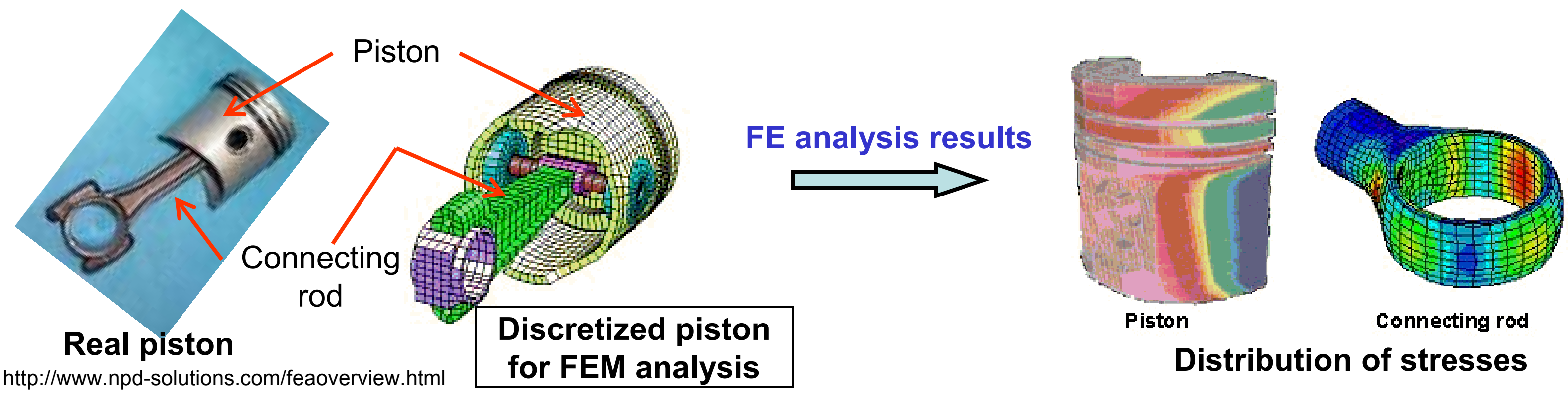
Solution of Simultaneous Equations Using Matrix Algebra

A vital tool for solving very large number of simultaneous equations

Why huge number of simultaneous equations in this type of analyses?

- Numerical analyses, such as the **finite element method** (FEM) and **finite difference method** (FDM) are two effective and powerful analytical tools for engineering analysis in in real but complex situations for:
  - Mechanical stress and deformation analyses of machines and structures
  - Thermofluid analyses for temperature distributions in solids, and fluid flow behavior requiring solutions in pressure drops and local velocity, as well as fluid-induced forces
- The essence of FEM and FDM is to **DISCRETIZE** "real structures" or "flow patterns" of complex configurations and loading/boundary conditions into FINITE number of sub-components (called elements) inter-connected at common NODES
- Analyses are performed in individual ELEMENTS instead of entire complex solids or flow patterns

- Example of discretization of a piston in an internal combustion engine and the results in stress distributions in piston and connecting rod:



- FEM or FDM analyses result in one algebraic equation for every NODE in the discretized model – Imagine the total number of (simultaneous) equations need to be solved !!
- Analyses using FEM requiring solutions of tens of thousands simultaneous equations are not unusual.

Solution of Simultaneous Equations Using Inverse Matrix Technique

Let us express the n-simultaneous equations to be solved in the following form:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots\dots + a_{1n}x_n = r_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots\dots + a_{2n}x_n = r_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots\dots\dots + a_{3n}x_n = r_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots\dots\dots + a_{mn}x_n = r_n
\end{aligned} \tag{8.16}
$$

where a_{11}, a_{12}, ………, a_{mn} are constant coefficients
x_1, x_2, …………., x_n are the unknowns to be solved
r_1, r_2, …………., r_n are the "resultant" constants

The n-simultaneous equations in Equation (8.16) can be expressed in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \bullet \\ \bullet \\ r_n \end{Bmatrix} \qquad (8.17)$$

or in an abbreviate form:

$$[A]\{x\} = \{r\} \qquad (8.18)$$

in which [A] = Coefficient matrix with m-rows and n-columns
{x} = Unknown matrix, a column matrix
{r} = Resultant matrix, a column matrix

Now, if we let $[A]^{-1}$ = the inverse matrix of [A], and multiply this $[A]^{-1}$ on both sides of Equation (8.18), we will get:

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{r\}$$

Leading to: $[I]\{x\} = [A]^{-1}\{r\}$,in which [I] = a unity matrix

The unknown matrix, and thus the values of the unknown quantities $x_1, x_2, x_3, \ldots, x_n$ may be obtained by the following relation:

$$\{x\} = [A]^{-1}\{r\} \qquad (8.19)$$

Example 8.6

Solve the following simultaneous equation using matrix inversion technique;

$$4x_1 + x_2 = 24$$
$$x_1 - 2x_2 = -21$$

Let us express the above equations in a matrix form:

$$[A]\{x\} = \{r\}$$

where $[A] = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$ and $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ *and* $\{r\} = \begin{Bmatrix} 24 \\ -21 \end{Bmatrix}$

Following the procedure presented in Section 8.5, we may derive the inverse matrix $[A]^{-1}$ to be:

$$[A]^{-1} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}$$

Thus, by using Equation (8.19), we have:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [A]^{-1}\{r\} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}\begin{Bmatrix} 24 \\ -21 \end{Bmatrix} = \frac{1}{9}\begin{Bmatrix} 2x24 - 1x21 = 27 \\ 1x24 + (-4)(-21) = 108 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 12 \end{Bmatrix}$$

from which we solve for $x_1 = 3$ and $x_2 = 12$

Solution of Simultaneous Equations Using Gaussian Elimination Method

Johann Carl Friedrich Gauss (1777 – 1855)

A German astronomer (planet orbiting),
Physicist (molecular bond theory, magnetic theory, etc.), and
Mathematician (differential geometry, Gaussion distribution in statistics
Gaussion elimination method, etc.)

- Gaussian elimination method and its derivatives, e.g., Gaussian-Jordan elimination method and Gaussian-Siedel iteration method are widely used in solving large number of simultaneous equations as required in many modern-day numerical analyses, such as FEM and FDM as mentioned earlier.

- The principal reason for Gaussian elimination method being popular in this type of applications is the formulations in the solution procedure can be readily programmed using concurrent programming languages such as FORTRAN for digital computers with high computational efficiencies

The essence of Gaussian elimination method:

1) To convert the square coefficient matrix [A] of a set of simultaneous equations into the form of "Upper triangular" matrix in Equation (8.5) using an "*elimination procedure*"

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{Via "elimination process}} [A]^{upper} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

2) The last unknown quantity in the converted upper triangular matrix in the simultaneous equations becomes immediately available.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \end{Bmatrix} \Longrightarrow x_3 = r_3''/a_{33}''$$

3) The second last unknown quantity may be obtained by substituting the newly found numerical value of the last unknown quantity into the second last equation:

$$a_{22}' x_2 + a_{23}' x_3 = r_2' \Longrightarrow x_2 = \frac{r_2' - a_{23}' x_3}{a_{22}'}$$

4) The remaining unknown quantities may be obtained by the similar procedure, which is termed as "*back substitution*"

The Gaussian Elimination Process:

We will demonstrate this process by the solution of 3-simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2 \qquad \text{(8.20 a,b,c)}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3$$

We will express Equation (8.20) in a matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \qquad \text{8.21)}$$

or in a simpler form: $[A]\{x\} = \{r\}$

We may express the unknown x_1 in Equation (8.20a) in terms of x_2 and x_3 as follows:

$$x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3$$

Now, if we substitute x1 in Equation (8.20b and c) by $x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3$

we will turn Equation (8.20) from:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1 \rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2 \rightarrow 0 + \left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{23} - a_{21}\frac{a_{13}}{a_{11}}\right)x_3 = r_2 - \frac{a_{21}}{a_{11}}r_1 \quad (8.22)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3 \rightarrow 0 + \left(a_{32} - a_{31}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{33} - a_{31}\frac{a_{13}}{a_{11}}\right)x_3 = r_3 - \frac{a_{31}}{a_{11}}r_1$$

You do not see x_1 in the new Equation (20b and c) anymore –
So, x_1 is "eliminated" in these equations after Step 1 elimination

The new matrix form of the simultaneous equations has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a^1_{22} & a^1_{23} \\ 0 & a^1_{32} & a^1_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r^1_2 \\ r^1_3 \end{Bmatrix} \quad (8.23)$$

$$a^1_{22} = a_{22} - a_{21}\frac{a_{12}}{a_{11}} \qquad a^1_{23} = a_{23} - a_{21}\frac{a_{13}}{a_{11}}$$

$$a^1_{32} = a_{32} - a_{31}\frac{a_{12}}{a_{11}} \qquad a^1_{33} = a_{33} - a_{31}\frac{a_{13}}{a_{11}}$$

$$r^1_2 = r_2 - \frac{a_{21}}{a_{11}}r_1 \qquad r^1_3 = r_3 - \frac{a_{31}}{a_{11}}r_1$$

The index numbers ("1") indicates "elimination step 1" in the above expressions

Step 2 elimination involve the expression of x_2 in Equation (8.22b) in term of x_3:

from
$$0+\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)x_2+\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3=r_2-\frac{a_{21}}{a_{11}}r_1 \qquad (8.22b)$$

to
$$x_2=\frac{r_2-\frac{a_{21}}{a_{11}}r_1-\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3}{\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)}$$

and submitted it into Equation (8.22c), resulting in eliminate x_2 in that equation.

The matrix form of the original simultaneous equations now takes the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{2} & a_{23}^{2} \\ 0 & 0 & a_{33}^{2} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} r_1^{2} \\ r_2^{2} \\ r_3^{2} \end{Bmatrix} \qquad (8.24)$$

We notice the coefficient matrix [A] now has become an “upper triangular matrix,” from which we have the solution
$$x_3=\frac{r_3^{2}}{a_{33}^{2}}$$

The other two unknowns x2 and x1 may be obtained by the “back substitution process from Equation (8.24),such as:
$$x_2=\frac{r_2^{2}-a_{23}^{2}x_3}{a_{22}^{2}}=\frac{r_2^{2}-a_{23}^{2}\frac{r_3^{2}}{a_{33}^{2}}}{a_{22}^{2}}$$

Recurrence relations for Gaussian elimination process:

Given a general form of n-simultaneous equations:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}x_n = r_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}x_n = r_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{3n}x_n = r_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{mn}x_n = r_n
\end{aligned}
\quad (8.16)
$$

The following recurrence relations can be used in Gaussian elimination process:

For elimination:

i > n and j>n

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1}\frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \quad (8.25a)$$

$$r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1}\frac{r_{n}^{n-1}}{a_{nn}^{n-1}} \quad (8.25b)$$

For back substitution

$$x_i = \frac{r_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}} \quad with \quad i = n-1, n-2, \ldots\ldots, 1 \quad (8.26)$$

Example

Solve the following simultaneous equations using Gaussian elimination method:

$$\begin{aligned} x + z &= 1 \\ 2x + y + z &= 0 \\ x + y + 2z &= 1 \end{aligned} \qquad \text{(a)}$$

Express the above equations in a matrix form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \qquad \text{(b)}$$

If we compare Equation (b) with the following typical matrix expression of 3-simultaneous equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

we will have the following:

$a_{11} = 1$ $a_{12} = 0$ $a_{13} = 1$

$a_{21} = 2$ $a_{22} = 1$ $a_{23} = 1$ and

$a_{31} = 1$ $a_{32} – 1$ $a_{33} = 2$

$r_1 = 1$

$r_2 = 0$

$r_3 = 1$

Let us use the recurrence relationships for the elimination process in Equation (8.25):

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1}\frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \qquad r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1}\frac{r_{n}^{n-1}}{a_{nn}^{n-1}}$$

with I >n and j> n

Step 1 n = 1, so i = 2,3 and j = 2,3

For i = 2, j = 2 and 3:

i = 2, j = 2: $$a_{22}^{1} = a_{22}^{0} - a_{21}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{12} - a_{21}\frac{a_{12}}{a_{11}} = 1 - 2\frac{0}{1} = 1$$

i = 2, j = 3: $$a_{23}^{1} = a_{23}^{0} - a_{21}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{23} - a_{21}\frac{a_{13}}{a_{11}} = 1 - 2\frac{1}{1} = -1$$

i = 2: $$r_{2}^{1} = r_{2}^{o} - a_{21}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{2} - a_{21}\frac{r_{1}}{a_{11}} = 0 - 2\frac{1}{1} = -2$$

For i = 3, j = 2 and 3:

i = 2, j = 2: $$a_{32}^{1} = a_{32}^{0} - a_{31}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{32} - a_{31}\frac{a_{12}}{a_{11}} = 1 - 1\frac{0}{1} = 1$$

i = 3, j = 3: $$a_{33}^{1} = a_{33}^{0} - a_{31}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{33} - a_{31}\frac{a_{13}}{a_{11}} = 2 - 1\frac{1}{1} = 1$$

i = 3: $$r_{3}^{1} = r_{3}^{0} - a_{31}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{3} - a_{31}\frac{r_{1}}{a_{11}} = 1 - 1\frac{1}{1} = 0$$

So, the original simultaneous equations after Step 1 elimination have the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & a_{32}^1 & a_{33}^1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^1 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 0 \end{Bmatrix}$$

We now have: $a_{21}^1 = 0 \quad a_{22}^1 = 1 \quad a_{23}^1 = -1$

$a_{31}^1 = 0 \quad a_{32}^1 = 1 \quad a_{33}^1 = 1$

$r_2^1 = -2 \quad r_3^1 = 0$

Step 2 n = 2, so i = 3 and j = 3 (i > n, j > n)

i = 3 and j = 3: $$a_{33}^2 = a_{33}^1 - a_{32}^1 \frac{a_{23}^1}{a_{22}^1} = 1 - 1\frac{(-1)}{1} = 2$$

$$r_3^2 = r_3^1 - a_{32}^1 \frac{r_2^1}{a_{22}^1} = 0 - 1\frac{(-2)}{1} = 2$$

The coefficient matrix [A] has now been triangularized, and the original simultaneous equations has been transformed into the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^2 \end{Bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 2 \end{Bmatrix}$$

We get from the last equation with (0) x_1 + (0) x_2 + 2 x_3 = 2, from which we solve for x_3 = 1. The other two unknowns x2 and x1 can be obtained by back substitution of x_3 using Equation (8.26):

$$x_2 = \left(r_2 - \sum_{j=3}^{3} a_{2j} x_j \right) / a_{22} = (r_2 - a_{23} x_2) / a_{22} = [-2 - (-1)(1)] / 1 = -1$$

and $$x_1 = \left(r_1 - \sum_{j=2}^{3} a_{1j} x_j \right) / a_{11} = [r_1 - (a_{12} x_2 + a_{13} x_3)] / a_{11}$$

$$= \{1 - [0(-1) + 1(1)]\} / 1 = 0$$

We thus have the solution: x = x_1 = 0; y = x_2 = -1 and z = x_3 = 1