

ME 130 Applied Engineering Analysis

Chapter 6

Review of Fourier Series and Its Applications in Mechanical Engineering Analysis

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Chapter Outline

- **Introduction**
- **Mathematical Expressions of Fourier Series**
- **Application in engineering analysis**
- **Convergence of Fourier Series**

Introduction



Jean Baptiste Joseph Fourier

1749-1829

A French mathematician

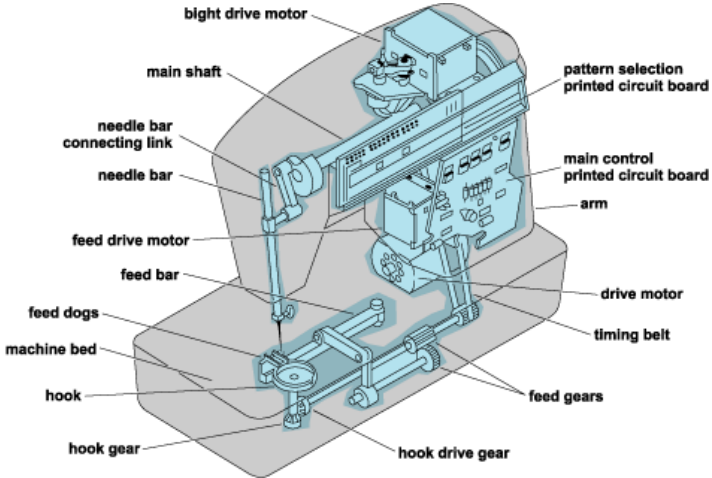
Major contributions to engineering analysis:

- Mathematical theory of heat conduction
(Fourier law of heat conduction in Chapter 3)
- Fourier series representing periodical functions
- Fourier transform
Similar to Laplace transform, but for transforming variables in the range of $(-\infty$ and $+\infty)$
- a powerful tool in solving differential equations

Periodic Physical Phenomena:



Motions of ponies

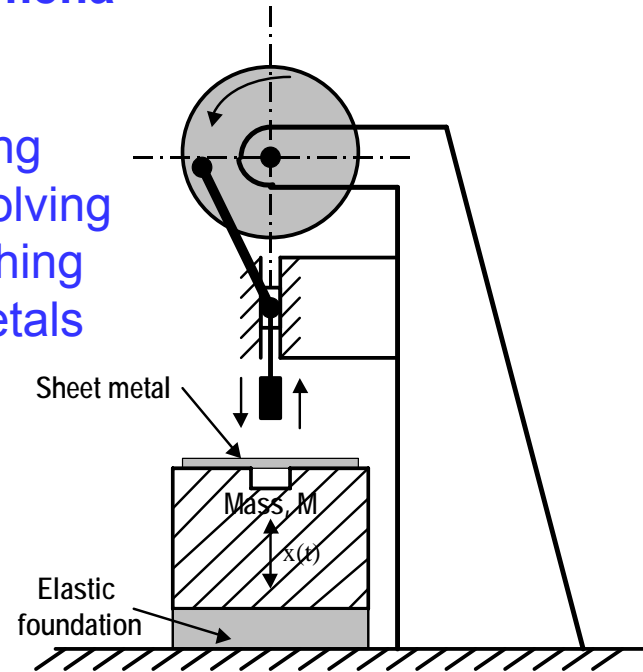


Forces on the needle

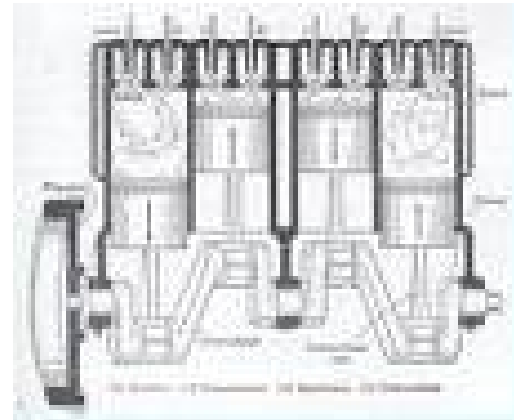
Machines with Periodic Physical Phenomena



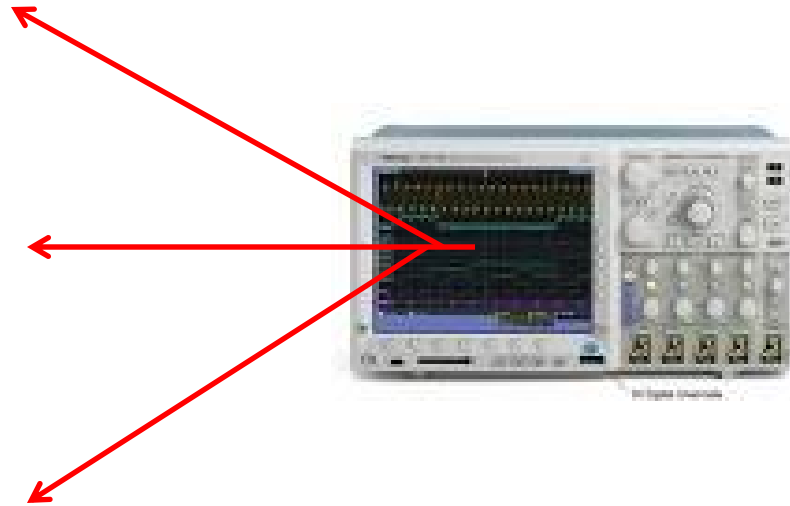
A stamping machine involving cyclic punching of sheet metals



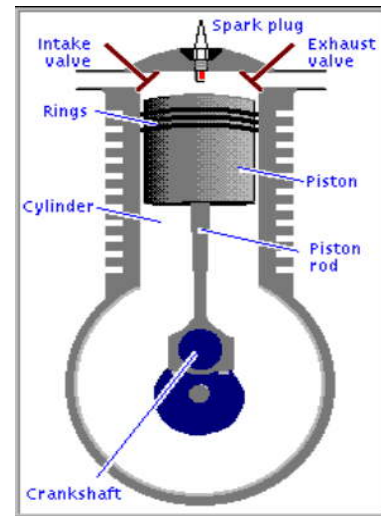
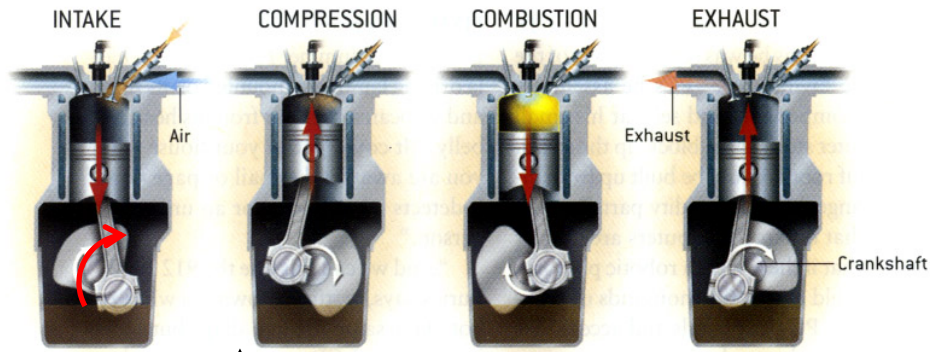
In a 4-stroke internal combustion engine:
Cyclic gas pressures on cylinders,
and forces on connecting rod and crank shaft



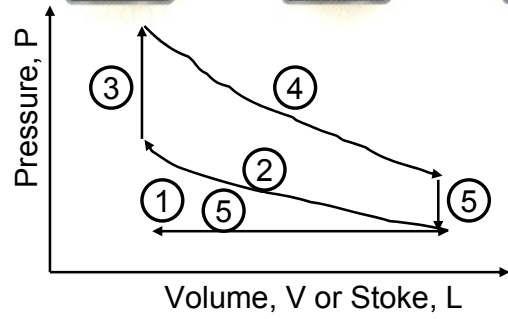
Mathematical expressions for periodical signals from an oscilloscope by Fourier series:



The periodic variation of gas pressure in a 4-stroke internal combustion engine:

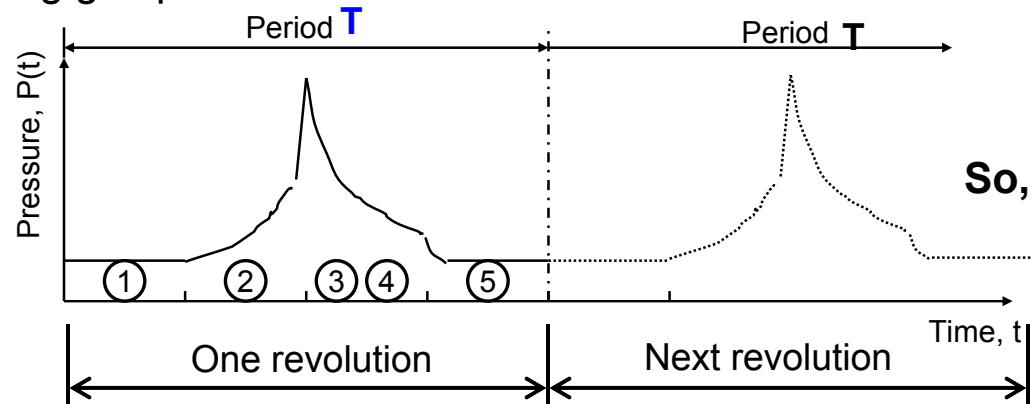


The P-V Diagram
 P = gas pressure
 in cylinders



- ① - Intake
- ② - Compression
- ③ - Combustion
- ④ - Expansion
- ⑤ - Exhaust

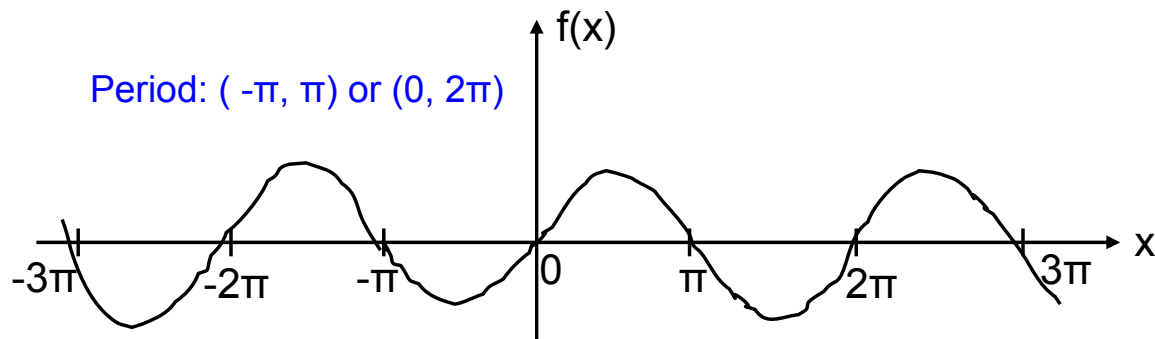
But the stroke l varies with time of the rotating crank shaft, so the time-varying gas pressure is illustrated as:



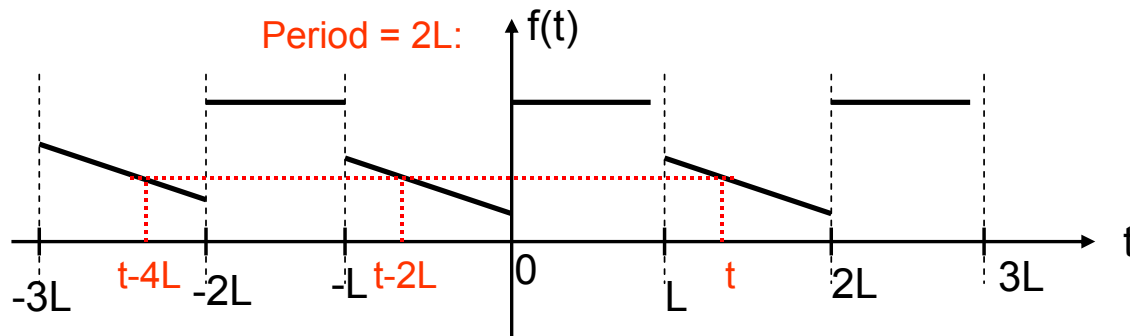
So, $P(t)$ is a periodic function with period T

FOURIER SERIES – The mathematical representation of periodic physical phenomena

- Mathematical expression for periodic functions:
 - If $f(x)$ is a periodic function with variable x in **ONE period $2L$**
 - Then $f(x) = f(x \pm 2L) = f(x \pm 4L) = f(x \pm 6L) = f(x \pm 8L) = \dots = f(x \pm 2nL)$ where $n = \text{any integer number}$



(a) Periodic function with period $(-\pi, \pi)$



(b) Periodic function with period $(-L, L)$

Mathematical Expressions of Fourier Series

- Required conditions for Fourier series:
 - The mathematical expression of the periodic function $f(x)$ in one period must be available
 - The function in one period is defined in an interval ($c < x < c+2L$) in which $c = 0$ or any arbitrarily chosen value of x , and $L =$ half period
 - The function $f(x)$ and its first order derivative $f'(x)$ are either continuous or piece-wise continuous in $c < x < c+2L$
- The mathematical expression of Fourier series for periodic function $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = f(x \pm 2L) = f(x \pm 4L) = \dots \quad (6.1)$$

where a_0 , a_n and b_n are **Fourier coefficients**, to be determined by the following integrals:

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx \quad n = 0, 1, 2, 3, \dots \quad (6.2a)$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (6.2b)$$

Example 6.1

Derive a Fourier series for a periodic function with period $(-\pi, \pi)$:

We realize that the period of this function $2L = \pi - (-\pi) = 2\pi$

The half period is $L = \pi$

If we choose $c = -\pi$, we will have $c+2L = -\pi + 2\pi = \pi$

Thus, by using Equations (6.1) and (6.2), we will have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \operatorname{Cos} \frac{n\pi x}{L=\pi} + b_n \operatorname{Sin} \frac{n\pi x}{L=\pi} \right)$$

and

$$a_n = \frac{1}{L=\pi} \int_{c=-\pi}^{c+2L=-\pi+2\pi} f(x) \operatorname{Cos} \frac{n\pi x}{L=\pi} dx$$

$$b_n = \frac{1}{L=\pi} \int_{c=-\pi}^{c+2L=-\pi+2\pi} f(x) \operatorname{Sin} \frac{n\pi x}{L=\pi} dx$$

Hence, the Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \operatorname{Cos}(nx) + b_n \operatorname{Sin}(nx)) \tag{6.3}$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{Cos}(nx) dx \quad n = 0, 1, 2, 3, \dots \tag{6.4a}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{Sin}(nx) dx \quad n = 1, 2, 3, \dots \tag{6.4b}$$

We notice the period $(-\pi, \pi)$ might not be practical, but it appears to be common in many applied math textbooks. Here, we treat it as a special case of Fourier series.

Example 6.2

Derive a Fourier series for a periodic function $f(x)$ with a period $(-\ell, \ell)$

Let us choose $c = -\ell$, and the period $2L = \ell - (-\ell) = 2\ell$, and the half period $L = \ell$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L = \ell} + b_n \sin \frac{n\pi x}{L = \ell} \right)$$

$$a_n = \frac{1}{L = \ell} \int_{c=-\ell}^{c+2L=-\ell+2\ell} f(x) \cos \frac{n\pi x}{L = \ell} dx$$

$$b_n = \frac{1}{L = \ell} \int_{c=-\ell}^{c+2L=-\ell+2\ell} f(x) \sin \frac{n\pi x}{L = \ell} dx$$

Hence the Fourier series of the periodic function $f(x)$ becomes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \tag{6.5}$$

with

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx \quad n = 0, 1, 2, 3, \dots \tag{6.6a}$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx \quad n = 1, 2, 3, \dots \tag{6.6b}$$

Example 6.3

Derive a Fourier series for a periodic function $f(x)$ with a period $(0, 2L)$.

As in the previous examples, we choose $c = 0$, and half period to be L . We will have the Fourier series in the following form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{c=0}^{c+2L=-0+2L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{c=0}^{c+2L=0+2L} f(x) \sin \frac{n\pi x}{L} dx$$

The corresponding Fourier series thus has the following form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (6.7)$$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x) dx \quad (6.8a)$$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (6.8b)$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (6.8c)$$

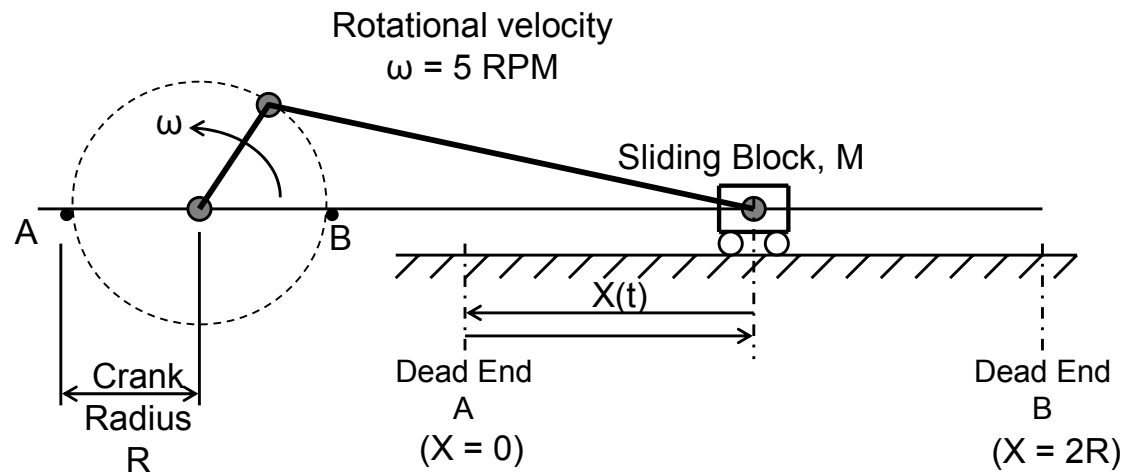
Periodic functions with periods $(0, 2L)$ are more realistic. Equations (6.7) and (6.8) are thus more practical in engineering analysis.

The class is encouraged to study Examples 6.4 and 6.5

Example (Problem 6.4 and Problem (3) of Final exam S09)

Derive a function describing the position of the sliding block M in one period in a slide mechanism as illustrated below. If the crank rotates at a constant velocity of 5 rpm.

- (a) Illustrate the periodic function in three periods, and
- (b) Derive the appropriate Fourier series describing the position of the sliding block $x(t)$ in which t is the time in minutes



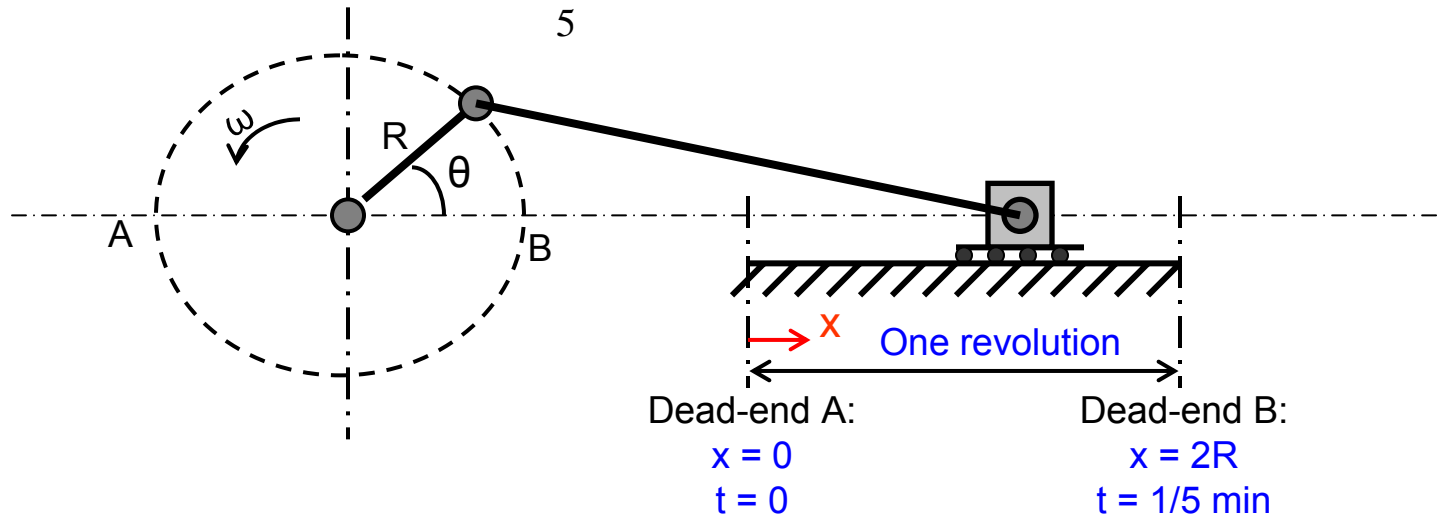
Solution:

(a) Illustrate the periodic function in three periods:

Determine the angular displacement of the crank:

We realize the relationship: $\text{rpm } N = \omega/(2\pi)$, and $\theta = \omega t$, where ω = angular velocity and θ = angular displacement relating to the position of the sliding block

For $N = 5$ rpm, we have: $\frac{\theta}{2\pi} = \frac{t}{\frac{1}{5}} = 5t$ Based on one revolution ($\theta=2\pi$) corresponds to $1/5$ min. We thus have $\theta = 10\pi t$



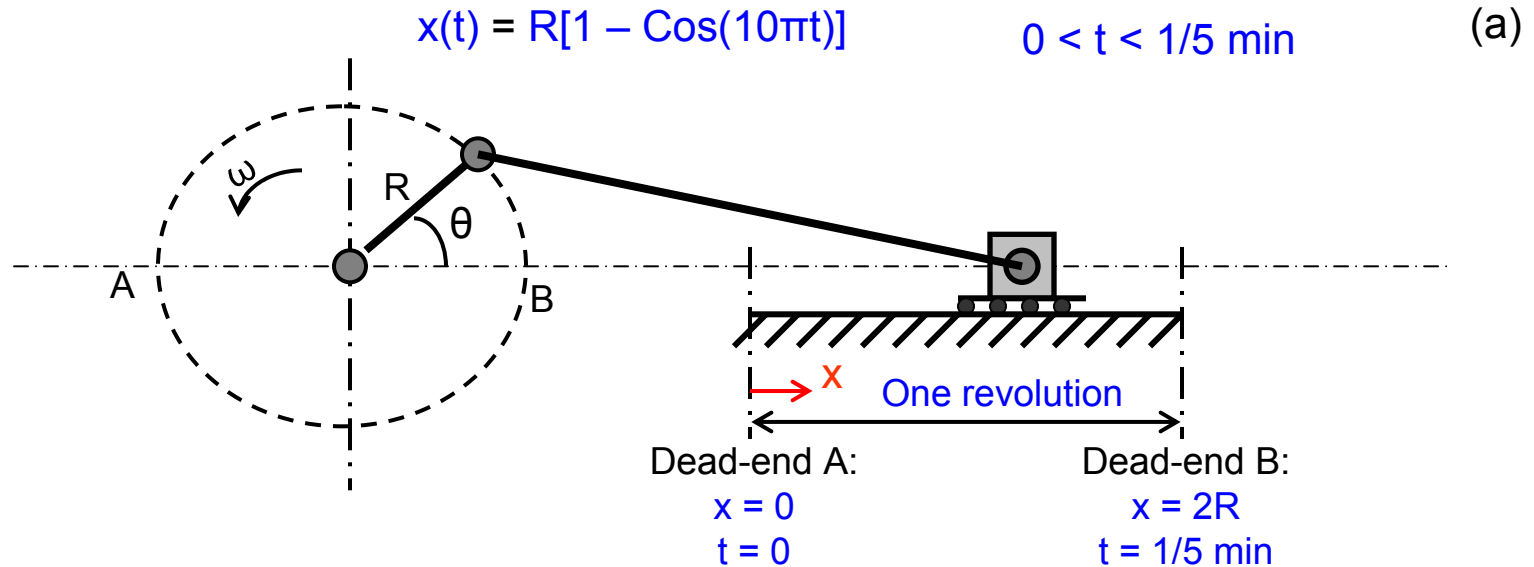
Position of the sliding block along the x -direction can be determined by:

$$x = R - R\cos\theta$$

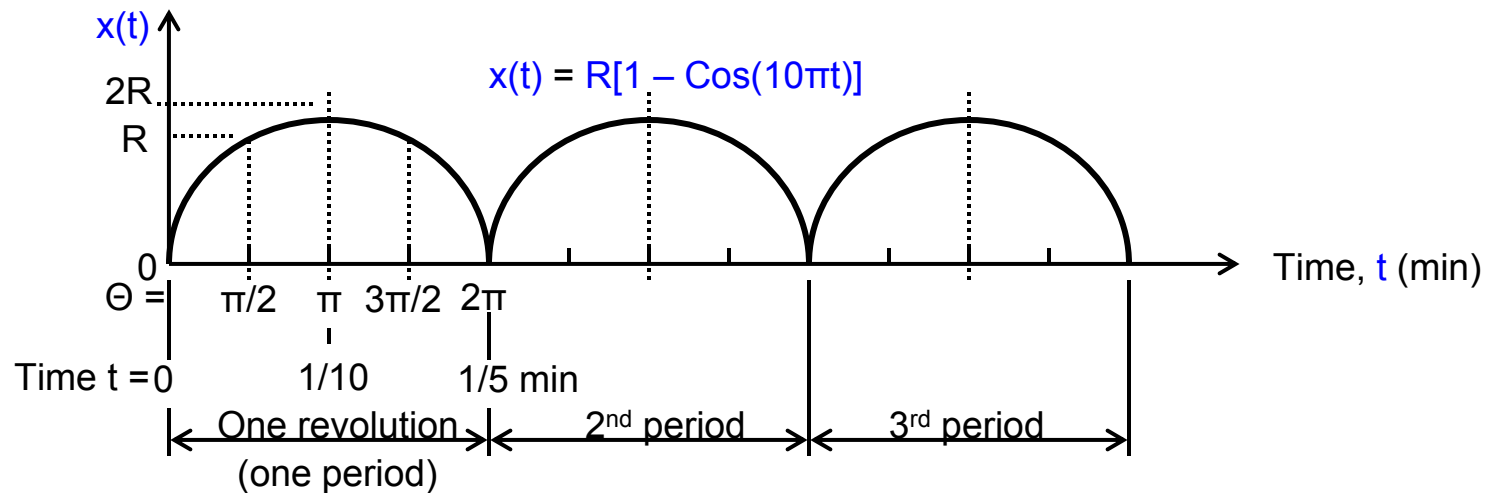
or

$$x(t) = R - R\cos(10\pi t) = R[1 - \cos(10\pi t)] \quad 0 < t < 1/5 \text{ min}$$

We have now derived the periodic function describing the instantaneous position of the sliding block as:



Graphical representation of function in Equation (a) can be produced as:



(b) Formulation of Fourier Series:

We have the periodic function: $x(t) = R[1 - \text{Cos}(10\pi t)]$ with a period: $0 < t < 1/5 \text{ min}$

If we choose $c = 0$ and period $2L = 1/5$, we will have the Fourier series expressed in the following form by using Equations (6.7) and (6.8):

$$\begin{aligned} x(t) &= \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \text{Cos} \frac{n\pi t}{L=1/10} + b_n \text{Sin} \frac{n\pi t}{L=1/10} \right] \\ &= \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \text{Cos} 10n\pi t + b_n \text{Sin} 10n\pi t] \end{aligned} \quad (b)$$

with
$$a_n = \frac{1}{1/10} \int_0^{1/5} x(t) \text{Cos} 10n\pi t dt = -\frac{R}{2\pi} \left[\frac{\text{Sin} 2(1-n)\pi}{1-n} + \frac{\text{Sin} 2(1+n)\pi}{1+n} \right] \quad (c)$$

We may obtain coefficient a_o from Equation (c) to be $a_o = 0$:

The other coefficient b_n can be obtained by:

$$\begin{aligned} b_n &= 10 \int_0^{1/5} x(t) \text{Sin} 10n\pi t dt = 10 \int_0^{1/5} R(1 - \text{Cos} 10\pi t) \text{Sin} 10n\pi t dt \\ &= \frac{R}{2(n-1)\pi} [\text{Cos} 2(n-1)\pi - 1] + \frac{R}{2(n+1)\pi} [\text{Cos} 2(n+1)\pi - 1] \end{aligned} \quad (d)$$

Convergence of Fourier Series

We have learned the mathematical representation of periodic functions by Fourier series
In Equation (6.1):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = f(x \pm 2L) = f(x \pm 4L) = \dots \quad (6.1)$$

This form requires the summation of “INFINITE” number of terms, which is UNREALISTIC.

The question is “HOW MANY” terms one needs to include in the summation in order to reach an accurate representation of the periodic function?

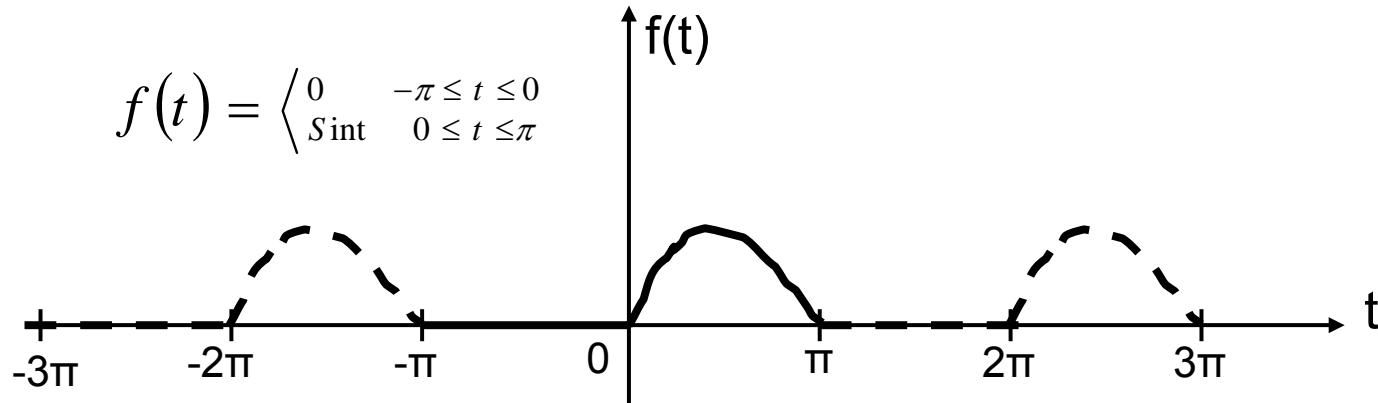
The following example will give some idea on the relationship of the “number of terms in the Fourier series” and the “accurate representation of the periodic function”:

Example 6.6

Derive the Fourier series for the following periodic function:

$$f(t) = \begin{cases} 0 & -\pi \leq t \leq 0 \\ \sin t & 0 \leq t \leq \pi \end{cases}$$

This function can be graphically represented as:



We identified the period to be: $2L = \pi - (-\pi) = 2\pi$, and from Equation (6.3), we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \text{Cos}(nx) + b_n \text{Sin}(nx)) \quad (\text{a})$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \text{Cos}(nt) dt = \frac{1}{\pi} \int_{-\pi}^0 (0) \text{Cos} nt dt + \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Cos} nt dt = \frac{1 + \text{Cos} n\pi}{(1 - n^2)\pi} \quad \text{for } n \neq 1$$

and

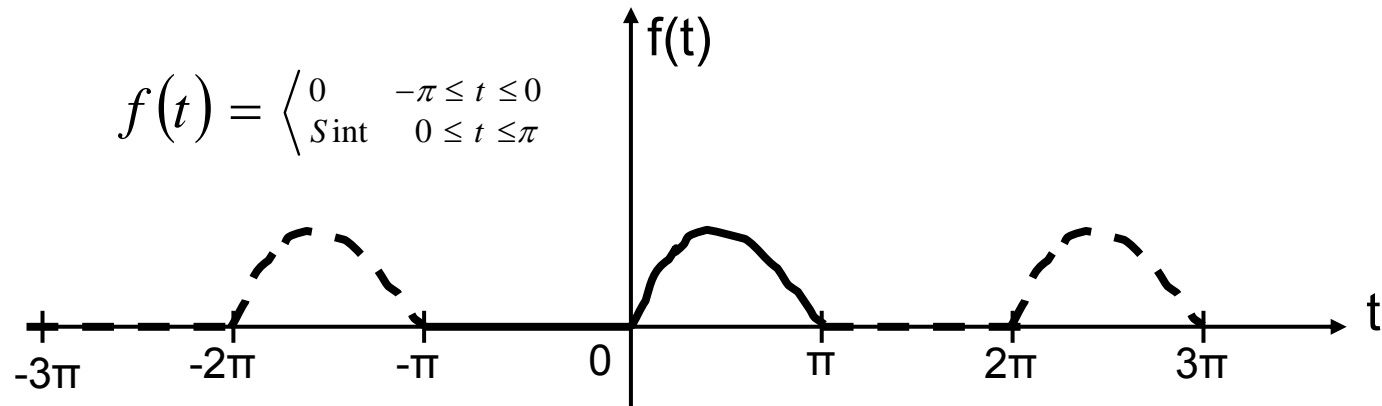
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \text{Sin}(nt) dt = \frac{1}{\pi} \int_{-\pi}^0 (0) \text{Sin} nt dt + \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Sin} nt dt \quad n = 1, 2, 3, \dots$$

or

$$b_n = \frac{1}{\pi} \left\{ \frac{1}{2} \left[\frac{\text{Sin}(1-n)t}{1-n} - \frac{\text{Sin}(1+n)t}{1+n} \right] \right\} \Bigg|_0^{\pi} = 0 \quad \text{for } n \neq 1$$

For the case $n = 1$, the two coefficients become:

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Cost} dt = \frac{\text{Sin}^2 t}{2\pi} \Bigg|_0^{\pi} = 0 \quad \text{and} \quad b_1 = \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Sint} dt = \frac{1}{2}$$



The Fourier series for the periodic function with the coefficients become:

$$f(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} + \sum_{n=2}^{\infty} (a_n \text{Cos } nt + b_n \text{Sin } nt) \quad (b)$$

The Fourier series in Equation (b) can be expanded into the following infinite series:

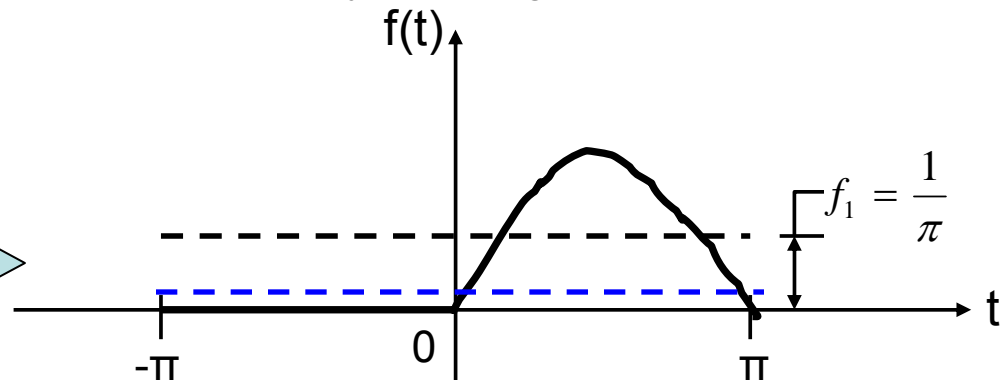
$$f(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} - \frac{2}{\pi} \left(\frac{\text{Cos } 2t}{3} + \frac{\text{Cos } 4t}{15} + \frac{\text{Cos } 6t}{35} + \frac{\text{Cos } 8t}{63} + \dots \right) \quad (c)$$

Let us now examine what the function would look like by including different number of terms in expression (c):

Case 1: Include only one term:

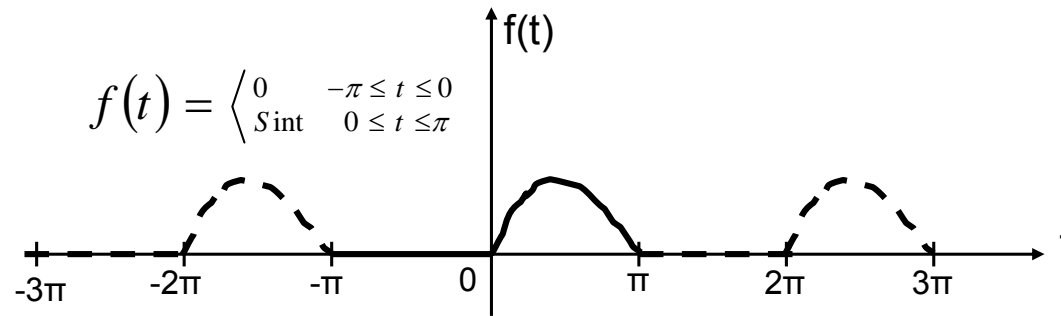
$$f(x) = f_1 = \frac{1}{\pi}$$

Graphically it will look like



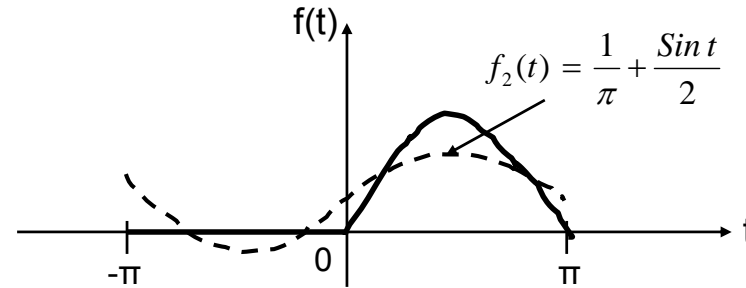
Observation: Not even closely resemble

- The Fourier series with one term does not converge to the function!



Case 2: Include 2 terms in Expression (b):

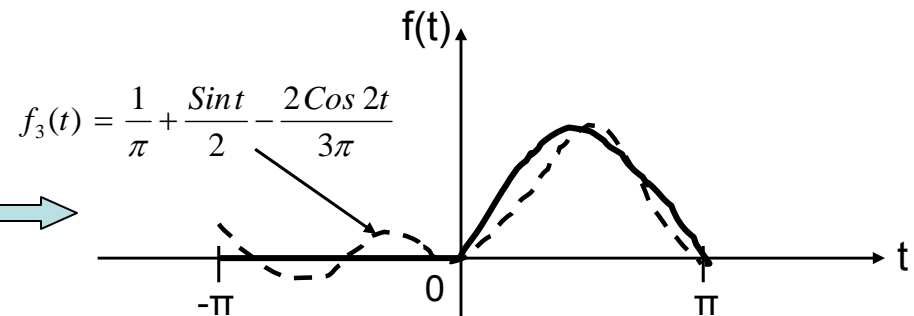
$$f(t) = f_2(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} \quad \Rightarrow$$



Observation: A Fourier series with 2 terms has shown improvement in representing the function

Case 3: Include 3 terms in Expression (b):

$$f(t) = f_3(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} - \frac{2\text{Cos } 2t}{3\pi} \quad \Rightarrow$$



Observation: A Fourier series with 3 terms represent the function much better than the two previous cases with 1 and 2 terms.

Conclusion: Fourier series converges better to the periodic function with more terms included in the series.

Practical consideration: It is not realistic to include infinite number of terms in the Fourier series for complete convergence. Normally an approach with 20 terms would be sufficiently accurate in representing most periodic functions

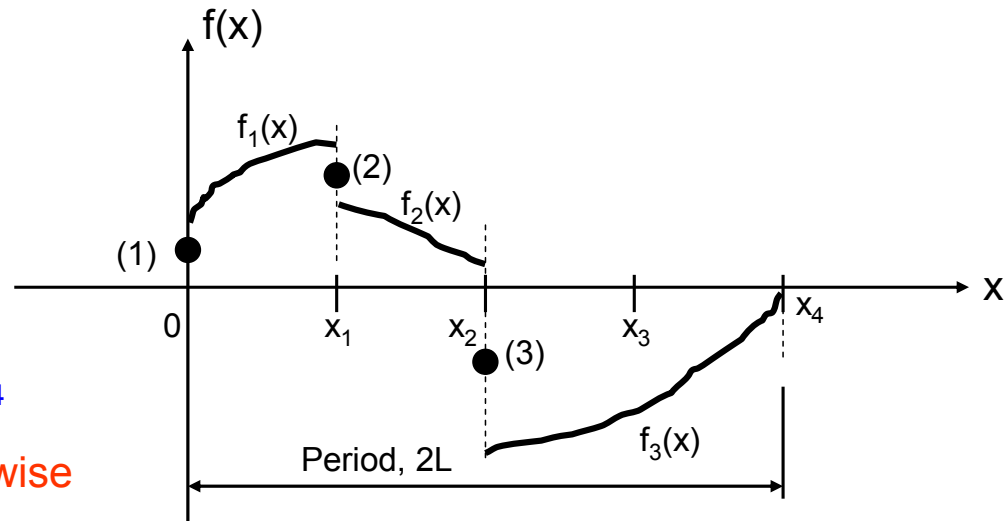
Convergence of Fourier Series at Discontinuities of Periodic Functions

Fourier series in Equations (6.1) to (6.3) converges to periodic functions everywhere except at discontinuities of piece-wise continuous function such as:

$$f(x) = \begin{cases} = f_1(x) & 0 < x < x_1 \\ = f_2(x) & x_1 < x < x_2 \\ = f_3(x) & x_2 < x < x_4 \end{cases}$$

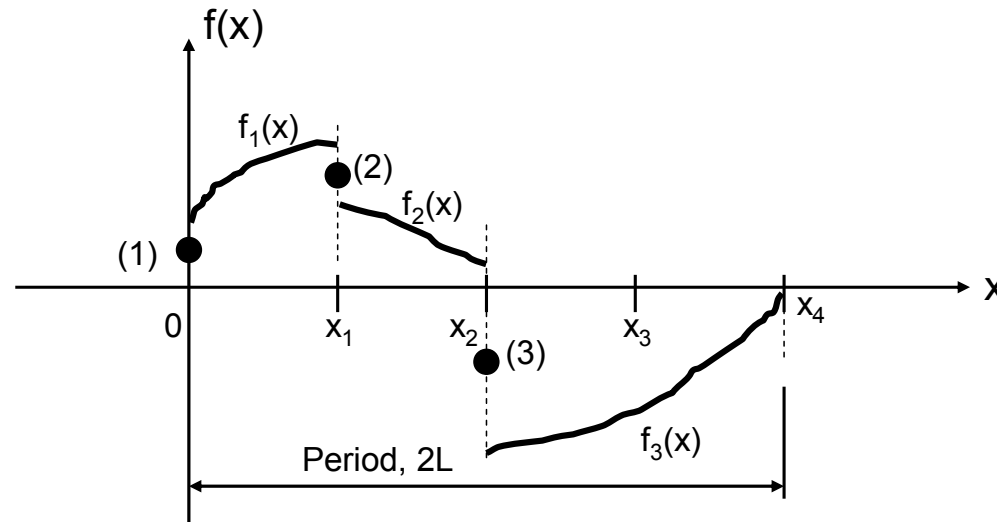
The periodic function $f(x)$ has discontinuities at: x_0, x_1, x_2 and x_4

The Fourier series for this piece-wise continuous periodic function will NEVER converge at these discontinuous points even with ∞ number of terms



- The Fourier series in Equations (6.1), (6.2) and (6.3) will converge every where to the function except these discontinuities, at which the series will converge **HALF-WAY** of the function values at these discontinuities.

Convergence of Fourier Series at Discontinuities of Periodic Functions



Convergence of Fourier series at HALF-WAY points:

$$f(0) = \frac{1}{2} f_1(0) \quad \text{at Point (1)}$$

$$f(x_1) = \frac{1}{2} [f_1(x_1) + f_2(x_1)] \quad \text{at Point (2)}$$

$$f(x_2) = \frac{1}{2} [f_2(x_2) + f_3(x_2)] \quad \text{at Point (3)}$$

$$f(x_4) = f_3(x_4) = \frac{1}{2} f_1(0) \quad \text{same value as Point (1)}$$