ME 130 Applied Engineering Analysis

Chapter 5

Review of Laplace Transform and Its Applications in Mechanical Engineering Analysis

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Chapter Outline

- Introduction
- Mathematical expression of Laplace transform of functions
- Properties of Laplace transform
- Inverse Laplace transform
- Laplace transform of derivatives
- Solution of differential equations by Laplace transform

Introduction

Laplace, Pierre-Simon (1749-1829)
- a French mathematician, astronomer and statistician.

Major accomplishments in mathematics:

- Laplace equation for electrical and mechanical potentials

$$\frac{\partial^2 P(x,y)}{\partial x^2} + \frac{\partial^2 P(x,y)}{\partial y^2} = 0$$

where P(x,y) = Temperature for thermal potential or electric charge

- Laplacian differential operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$, and

- Laplace transform:

$$L_x[F(x)] = \int_0^\infty e^{-sx} F(x)dx$$

or $$L_t[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

where F(x) is a function of variable x and f(t) is a function of variable t, and s = parameter

Laplace Transform in Engineering Analysis

- Laplace transforms is a mathematical operation that is used to "transform" a *variable (such as x, or y, or z, or t)* to a *parameter (s)*. Mathematically, it can be expressed as:

$$L_t\left[f(t)\right] = \int_0^\infty e^{-st}\, f(t)dt = F(s) \tag{5.1}$$

  where F(s) = expression of Laplace transform of function f(**t**) involving parameter s

- In a layman's term, Laplace transform is used to "transform" a variable in a function into a parameter

- So, after the transformation that variable is no longer a variable anymore, but should be treated as a "parameter", i.e a "constant under a specific condition"

- This "specific condition" for Laplace transform stipulates that:

  - *Laplace transform can only be used to transform variables that cover a range from "zero (**0**)" to infinity, (**∞**), that is:* $0 < t < \infty$

- *Any variable that does not vary within this range cannot be transformed using Laplace transform*
- Because time variable t is most common variable that varies from (0 to ∞), functions with variable t are commonly transformed by Laplace transform
- Lapalce transform is a valuable "tool" in solving:
  - Differential equations for example: electronic circuit equations, and
  - in "feedback control" systems for example, in stability and control of aircraft systems

Laplace transform of simple functions:

For $f(t) = t^2$ with $0 < t < \infty$:

$$L[f(t)] = \int_0^\infty e^{-st}\left(t^2\right)dt = e^{-st}\left[-\frac{2t^2}{2s} - \frac{2t}{s^2} - \frac{2}{s^3}\right]\Big|_0^\infty = \frac{2}{s^3} = F(s) \qquad \text{(a)}$$

For $f(t) = e^{at}$ with a = constant and $0 < t < \infty$:

$$L[f(t)] = \int_0^\infty e^{-st}\left(e^{at}\right)dt = \int_0^\infty e^{(-s+a)t}dt = \frac{1}{-s+a}e^{(-s+a)}\Big|_0^\infty = \frac{1}{s-a} = F(s) \qquad \text{(b)}$$

For $f(t) = Cos\omega t$ withω = constant and $0 < t < \infty$:

$$L[Cos\,\omega t] = \int_0^\infty e^{-st}(Cos\,\omega t)dt = \frac{e^{-st}}{(-s)^2+\omega^2}(-s\,Cos\,\omega t + \omega\,Sin\,\omega t)\Big|_0^\infty = \frac{s}{s^2+\omega^2} = F(s) \qquad \text{(c)}$$

Appendix 1 of the printed notes provides a Table of Laplace transforms of simple functions

For example, L[f(t)] of polynomial t^2 in Equation (a) is in Case 3 with n = 3 in the Table,
exponential function e^{at} in Equation (b) in Case 7, and
trigonometric function Cosωt in Equation (c) in Case 18

Properties of Laplace Transform

Laplace transform of functions by integration:

$$L_t\left[f(t)\right] = \int_0^\infty e^{-st}\, f(t)dt = F(s) \tag{5.1}$$

is not always easy.

Laplace transform (LT) Table in Appendix 1 is useful, but does not always have the required answer for the specific functions

- Following properties will be useful in finding the Laplace transform for specific functions:

1. Linear operators:

L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]

where a, b = constant coefficients

Example 5.4:

Find Laplace transform of function: $f(t) = 4t^2 - 3\text{Cos } t + 5e^{-t}$ with $0 < t < \infty$:

By using the linear operator, we may break up the transform into three individual transformations:

$L(4t^2 - 3\text{Cos } t + 5e^{-t}) = 4L[t^2] - 3L[\text{Cos } t] + 5L[e^{-t}] = F(s)$

Case 3 with n = 3 Case 18 with ω=1 Case 7 with a =-1 from the LT Table

Hence

$$F(s) = \frac{8}{s^3} - \frac{3s}{s^2+1} + \frac{5}{s+1}$$

Properties of Laplace Transform – Cont'd

2. Shifting property:

If the Laplace transform of a function, f(t) is **L[f(t)] = F(s)** by integration or from the Laplace Transform (LT) Table, then the Laplace transform of **$G(t) = e^{at}f(t)$** can be obtained by the following relationship:

$$L[G(t)] = L[e^{at}f(t)] = F(s-a) \qquad (5.6)$$

where a in the above formulation is the shifting factor, i.e. the parameter s in the transformed function f(t) has been shifted to (s-a) from s

Example 5.5:

Perform the Laplace transform on function: $F(t) = e^{2t}$ Sin(at), where a = constant

We may use the Laplace transform integral to get the solution, or we could get the solution by using the LT Table using the shifting property:

Since we can find $L[f(t)] = L[Sin\,at] = \frac{a}{s^2+a^2}$ (Case 17)

We may use the shifting property to get the Laplace transform of $F(t) = e^{2t}$ Sin(at), by

"shifting the parameter s by 2, or

$$L[F(t)] = L[e^{2t}\,Sin\,at] = \frac{a}{(s-2)^2+a^2}$$

3. Change of scale property:

If we know L[f(t)] = F(s) either from the LT Table, or by integral, we may find the Laplace transform of function f(a t) by the following expression:

$$L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad (5.7)$$

where a = scale factor for the change

Example 5.6:

Perform the Laplace transform of function F(t) = Sin3t.

Since we know the Laplace transform of f(t) = Sint from the LP Table as:

$$L[f(t)] = L[Sint] = \frac{1}{s^2+1} = F(s)$$

We may find the Laplace transform of F(t) using the Change scale property to be:

$$L[Sin3t] = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$$

Inverse Laplace Transform

We define the Laplace transform of a function f(t) to be:

Laplace transform

From here → to there

$$L_t\left[f(t)\right] = \int_0^\infty e^{-st} \; f(t)dt = F(s)$$

There are times we need to do:

$$L_t\left[f(t)\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = F(s)$$

to here ← From there

Inverse Laplace transform

There are **3** ways to inverse Laplace transform:

- Use LP Table by looking at F(s) in right column for corresponding f(t) in middle column - chance of success is not very good
- Use partial fraction method for F(s) = rational function (i.e. fraction functions involving polynomials), and
- The convolution theorem involving integrations

The Partial Fraction Method for Inverse Laplace Transform

- The expression of F(s) to be inversed should be in partial fractions as:

$$F(s) = \frac{P(s)}{Q(s)}$$

where polynomial P(s) is at least one order less than the order of polynomial Q(s)

- “Break” up the above rational function into summation of “simple fractions:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \ldots\ldots\ldots\ldots\ldots + \frac{A_n}{s-a_n} \tag{5.8}$$

where A_1; A_2,……..A_n, and a_1, a_2, ……….a_n are constants to be determined by comparing coefficients of terms on both sides of the equality:

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \ldots\ldots\ldots\ldots\ldots + \frac{A_n}{s-a_n}$$

- The inverse Laplace transform of P(s)/Q(s) becomes:

$$L^{-1}[F(s)] = \underset{\uparrow}{L^{-1}\left(\frac{P(s)}{Q(s)}\right)} = L^{-1}\left(\frac{A_1}{s-a_1}\right) + \underset{\uparrow}{L^{-1}\left(\frac{A_2}{s-a_2}\right)} + \ldots\ldots + L^{-1}\left(\frac{A_n}{s-a_n}\right)$$

A fraction ⟶ A sum of partial fractions

Example 5.7:

Perform the inverse Laplace transform of the function:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{3s+7}{s^2-2s-3}$$

Solution:

We may express F(s) in the following partial fraction form:

$$F(s) = \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$$

where A and B are constant coefficients

After expanding the above rational function and equating the terms in numerator:

3s + 7 = A(s+1) + B(s-3) = (A + B)s + (A – 3B)

We may solve for A and B from the simultaneous equations:

A + B = 3 and A – 3B = 7 yield A = 4 and B = -1

Thus we have:

$$\frac{3s+7}{s^2-2s-3} = \frac{4}{s-3} - \frac{1}{s+1}$$

The required Laplace transform is:

$$L^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right] = 4L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s+1}\right) = 4e^{3t} - e^{-t}$$

Example 5.8:

Perform the inverse Laplace transform:

$$L^{-1}[F(s)] = L^{-1}\left[\frac{P(s)}{Q(s)}\right] = L^{-1}\left[\frac{3s+1}{s^3 - s^2 + s - 1}\right]$$

Solution:

We may break up F(s) in the above expression in the form:

The polynomial in numerator is always one order less that in the denominator

$$\frac{P(s)}{Q(s)} = \frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

By following the same procedure, we have coefficients A = 2, B = -2 and C = 1, or:

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

We will thus have the inversed Laplace transform function f(t) to be:

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{3s+1}{s^3 - s^2 + s - 1}\right] = L^{-1}\left(\frac{2}{s-1}\right) - L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right) = 2e^t - 2Cos\,t + Sin\,t$$

Inverse Laplace Transform by Convolution Theorem

- This method involves the use of integration of expressions involving LT parameter s
- There is no restriction on the form of the expression of s – they can be rational functions, or trigonometric functions or exponential functions
- The convolution theorem works in the following way for inverse Laplace transform:

If we know the following:

$L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, with $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ and $G(s) = \int_0^\infty e^{-st} g(t)\,dt$

most likely from the LP Table

Then the desired inverse Laplace transformed: Q(s) = F(s) G(s) can be obtained by the following integrals:

$$L^{-1}[Q(s)] = L^{-1}[F(s)G(s)] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \qquad (5.9)$$

OR

$$L^{-1}[Q(s)] = L^{-1}[F(s)G(s)] = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \qquad (5.10)$$

Example 5.9:

Find the inverse of a Laplace transformed function with: $Q(s) = \dfrac{s}{\left(s^2 + a^2\right)^2}$

Solution:

We may express Q(s) in the following expression of the product of two functions:

$$Q(s) = \frac{s}{\left(s^2 + a^2\right)^2} = \frac{s}{s^2 + a^2} \cdot \frac{1}{s^2 + a^2}$$

It is our choice to select F(s) and G(s) from the above expression for the integrals in Equation (5.9) or (5.10).

Let us choose: $F(s) = \dfrac{s}{s^2 + a^2} \quad and \quad G(s) = \dfrac{1}{s^2 + a^2}$

From the LT Table, we have the following:

$$L^{-1}\left[F(s)\right] = Cos(at) = f(t) \quad and \quad L^{-1}\left[G(s)\right] = \frac{Sin(at)}{a} = g(t)$$

The inverse of Q(s) = F(s)G(s) is obtained by Equation (5.9) as:

$$L^{-1}\left[\frac{s}{\left(s^2 + a^2\right)^2}\right] = \int_0^t Cos\,a\tau \,\frac{Sin\,a(t-\tau)}{a}\,d\tau = \frac{t\,Sin\,at}{2a}$$

One will get the same result by using another convolution integral in Equation (5.10), or using partial fraction method in Equation (5.8)

Example 5.11:

Use convolution theorem to find the inverse Laplace transform: $Q(s) = \dfrac{1}{(s+1)(s^2+4)}$

We may express Q(s) in the following form:

$$Q(s) = \frac{1}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{1}{s^2+4} \qquad \text{(a)}$$

We choose F(s) and G(s) as:

$$F(s) = \frac{1}{s+1} = e^{-t} = f(t) \qquad \text{and} \qquad G(s) = \frac{1}{s^2+4} = \frac{1}{2} Sin\,2t = g(t)$$

Let us use Equation (5.10) for the inverse of Q(s) in Equation (a):

$$q(t) = \int_0^t F(t-\tau)\,G(\tau)\,d\tau = \int_0^t e^{-(t-\tau)}\left(\frac{1}{2} Sin\,2\tau\right)d\tau = \frac{1}{2}e^{-t}\int_0^t e^{\tau}\,Sin\,2\tau\;d\tau$$

After the integration:

$$q(t) = \frac{1}{2}e^{-t}\left[\frac{e^{\tau}\left(Sin\,2\tau - 2\,Cos\,2\tau\right)}{1+2^2}\right]\Bigg|_0^t = \frac{1}{10} Sin\,2t - \frac{1}{5} Cos\,2t + \frac{1}{5} e^{-t}$$

Laplace Transform of Derivatives

- We have learned the Laplace transform of function f(t) by:

$$L_t[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \qquad (5.1)$$

We realize the derivative of function f(t): $f'(t) = \frac{df(t)}{dt}$ is also a FUNCTION

So, there should be a possible way to perform the Laplace transform of the derivatives of functions, as long as its variable varies from zero to infinity.

- Laplace transform of derivatives is necessary steps in solving DEs using Laplace transform

- By following the mathematical expression for Laplace transform of functions shown in Equation (5.1), we have:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt \qquad (5.11)$$

Let: $\int_0^\infty udv = uv\Big|_0^\infty - \int_0^\infty vdu$

The above integration in Equation (5.11) can be performed by "Integration-by-parts:"

If we let:

$$\begin{array}{ccc} u = e^{-st} & and & dv = \left[\frac{df(t)}{dt}\right] \\ \Big\downarrow & & \Big\downarrow \\ du = -se^{-st}\,dt & & v = f(t) \end{array}$$

By substituting the above ‘u’, “du”, “dv” and “v” into the following:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt \tag{5.11}$$

$$\int_0^\infty udv = uv\Big|_0^\infty - \int_0^\infty vdu$$

We will have:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty e^{-st}\left[\frac{df(t)}{dt}\right]dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)\left(-se^{-st}\right)dt$$

leading to:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)\left(-se^{-st}\right)dt = -f(0) + s\int_0^\infty e^{-st} f(t)dt = -f(0) + sL[f(t)]$$

or in a simplified form:

$$L[f'(t)] = s\,L[f(t)] - f(0) \tag{5.12}$$

- Likewise, we may find the Laplace transform of second order derivative of function f(t) to be:

$$L[f''(t)] = s^2\,L[f(t)] - sf(0) - f'(0) \tag{5.13}$$

- A recurrence relation for Laplace transform of higher order (n) derivatives of function f(t) may be expressed as:

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots\ldots. f^{n-1}(0) \tag{5.14}$$

Example 5.12:

Find the Laplace transform of the second order derivative of function: f(t) = t Sint

The second order of derivative of f(t) meaning n = 2 in Equation (5.14), or as in Equation (5.13):

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0) \qquad (5.13)$$

We thus have:

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 L[f(t)] - sf(0) - f'(0)$$

Since $$f'(t) = \frac{df(t)}{dt} = \frac{d(t\,Sint)}{dt} = t\,Cost + Sint$$

We thus have:

$$L[f''(t)] = s^2 L[f(t)] - s\,f(0) - f'(0) = s^2 L[t\,Sint] - s(t\,Sint)\big|_{t=0} - (t\,Cost + Sint)\big|_{t=0}$$

$$= s^2\,L[t\,Sint]$$

Solution of DEs Using Laplace Transform

- One common application of Laplace transform is solving differential equations
- However, such application MUST satisfy the following two conditions:
  - The variable(s) in the function for the solution, e.g., x, y, z, t must cover the range of (0, ∞).
    That means the solution function, e.g., u(x) or u(t) MUST also be VALID for the range of (0, ∞)
  - ALL appropriate conditions for the differential equation MUST be available
- The solution procedure is as follows:

(1) Apply Laplace transform on EVERY term in the DE

(2) The Laplace transform of derivatives results in given conditions, such as f(0), f'(0), f"(0), etc. as shown in Equation (5.14)

(3) After apply the given values of the given conditions as required in Step (2), we will get an ALGEBRAIC equation for F(s) as defined in Equation (5.1):

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \tag{5.1}$$

(4) We thus can obtain an expression for F(s) from Step (3)

(5) The solution of the DE is the inverse of Laplace transformed F(s), i.e.,:

$$f(t) = L^{-1}[F(s)]$$

Example 5.13:

Solve the following DE with given conditions:

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t)=e^{-t}\,Sin\,t \qquad 0\le t\le\infty \tag{a}$$

where y(0) = 0 and y’(0) =1 (b)

Solution:

(1) Apply Laplace transform to EVERY term in the DE:

$$L\left[\frac{d^2y(t)}{dt^2}\right]+2L\left[\frac{dy(t)}{dt}\right]+5L[y(t)]=L\left[e^{-t}\,Sin\,t\right] \tag{c}$$

where $$L[y(t)]=\int_0^\infty y(t)\,e^{-st}\,dt=Y(s)$$

Use Equation (5.12) and (5.13) in Equation (c) will result in:

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+2\left[sY(s)-y(0)\right]+5Y(s)=\frac{1}{(s+1)^2+1}=\frac{1}{s^2+2s+2} \tag{d}$$

(2) Apply the given conditions in Equation (b) in Equation (d)

$$\left[s^2Y(s)-sy(\overset{=\,0}{0})-y'(\overset{=\,1}{0})\right]+2\left[sY(s)-y(\overset{=\,0}{0})\right]+5Y(s)=\frac{1}{(s+1)^2+1}=\frac{1}{s^2+2s+2}$$

(3) We can obtain the expression:

$$Y(s) = \frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)} \qquad \text{(e)}$$

(4) The solution of the DE in Equation (a) is the inverse Laplace transform of Y(s) in Equation (e), i.e. $y(t) = L^{-1}[Y(s)]$, or:

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}\right]$$

(5) The inverse Laplace transform of Y(s) in Equation (e) is obtained by using either "Partial fraction method" or "convolution theorem." The expression of Y(s) can be shown in the following form by "Partial fractions:"

$$Y(s) = \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5} = \frac{1}{3}\cdot\frac{1}{s^2+2s+2} + \frac{2}{3}\cdot\frac{1}{s^2+2s+5}$$

The inversion of Y9s) in the above form is:

$$y(t) = L^{-1}[Y(s)] = \frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

Leading to the solution of the DE in Equation (a) to be:

$$y(t) = L^{-1}[Y(s)] = \frac{1}{3}e^{-t}\ Sin\,t + \frac{2}{3}\,\frac{1}{2}e^{-t}\ Sin\,2t = \frac{1}{3}e^{-t}\left(Sin\,t + Sin\,2t\right)$$