ME 130 Applied Engineering Analysis

Chapter 4

Application of Second Order Differential Equations in Mechanical Engineering Analysis

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Chapter Outlines

- Review solution method of second order, homogeneous ordinary differential equations
- Applications in free vibration analysis
  - Simple mass-spring system
  - Damped mass-spring system
- Review solution method of second order, non-homogeneous ordinary differential equations
  - Applications in forced vibration analysis
  - Resonant vibration analysis
  - Near resonant vibration analysis
- Modal analysis

Part 1

Review Solution Method of Second Order, Homogeneous Ordinary Differential Equations

Typical form

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \tag{4.1}$$

where a and b in Equation (4.1) are constants

The solution of Equation (4.1) u(x) may be obtained by ASSUMING:

$$\mathbf{u(x) = e^{mx}} \tag{4.2}$$

in which **m** = constant to be determined

If the assumed solution u(x) in Equation (4.2) is valid solution, it must SATISFY the DE in Equation (4.1). That is:

$$\frac{d^2\left(e^{mx}\right)}{dx^2} + a\frac{d\left(e^{mx}\right)}{dx} + b\left(e^{mx}\right) = 0 \tag{a}$$

Because: $\frac{d^2\left(e^{mx}\right)}{dx^2} = m^2 e^{mx}$ and $\frac{d\left(e^{mx}\right)}{dx} = me^{mx}$

Upon substitution of the above into Equation (a) leading to: $m^2 e^{mx} + a\left(m e^{mx}\right) + b\left(e^{mx}\right) = 0$

Because e^{mx} in the expression cannot be zero (why?), we thus have:

$$\mathbf{m^2 + am + b = 0} \tag{4.3}$$

Equation (4.3) is a quadratic equation, and its solution for m are:

The quadratic equation: $m^2 + am + b = 0$

The TWO roots of the above quadratic equation have the forms:

$$m_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b} \qquad and \qquad m_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b} \tag{4.4}$$

This leads to two possible solutions for the function u(x) in Equation (4.1):

$$u(x) = c_1\, e^{m_1 x} + c_2\, e^{m_2 x} \tag{4.5}$$

where c_1 and c_2 are the TWO arbitrary constants to be determined by TWO specified conditions, and m_1 and m_2 are expressed in Equation (4.4)

Because the constant coefficients a and b in Equation (4.1) are fixed with the DE, the relative magnitudes of the a, b will result in significant forms in the solution in Equation (4.5) due to the "square root" parts in the expression of m_1 and m_2 in Equation (4.4). Square root of a negative number will lead to a complex number in the solution of the DE, which requires a special way of expressing it.

We thus need to look into 3 possible cases involving relative magnitudes a and b.

Case 1. $a^2 – 4b > 0$:

In such case, we realize that both m_1 and m_2 are real numbers. The solution of the Equation (4.1) is:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\; x/2} + c_2\, e^{-\sqrt{a^2 - 4b}\; x/2}\right) \qquad (4.6)$$

Case 2. $a^2 - 4b < 0$:

As described earlier, both these roots become complex numbers involving imaginary parts. The substitution of the m_1 and m_2 into Equation (4.5) will lead to the following:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1\, e^{\frac{ix}{2}\sqrt{4b - a^2}} + c_2 e^{-\frac{ix}{2}\sqrt{4b - a^2}}\right) \qquad (4.7)$$

in which . $i = \sqrt{-1}$

The complex form of the solution in Equation (4.7) is not always easily comprehended and manipulative in engineering analyses, a more commonly used form involving trigonometric functions are used:

$$u(x) = e^{-\frac{ax}{2}}\left[A\, Sin\left(\frac{1}{2}\sqrt{4b - a^2}\right)x + B\, Cos\left(\frac{1}{2}\sqrt{4b - a^2}\right)x\right] \qquad (4.8)$$

where A and B are arbitrary constants.

The expression in Equation (4.8) may be derived from Equation (4.7) using the Biot relation that has the form: . $e^{\pm i\theta} = Cos\theta \pm i\, Sin\theta$

Case 3. $a^2 - 4b = 0$:

Recall Equation (4.4):

$$m_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b} \qquad and \qquad m_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b}$$

The condition $a^2 - 4b = 0$ will lead to a situation: $m_1 = m_2 = a/2$ (b)

Substituting the m1 and m2 into Equation (4.5) will result in:

$$u(x) = (c_1 + c_2)e^{-\frac{a}{2}x} \quad or \quad u_1(x) = ce^{-\frac{a}{2}x}$$

with only ONE term with a constant in the solution, which is not complete for a 2nd order DE.

So, we will have to find the "missing" term in the solution u(x).

Realizing the fact that the assumed solution $u(x) = e^{mx}$ in Equation (4.2) results in one missing term, we need to find another assumed solution. Let us try the following:

$$\mathbf{u_2(x) = V(x)\, e^{mx}} \qquad (4.9)$$

where V(x) is a function of x, and it needs to be determined

We may follow the same procedure before in determining function V(x), that is the assumed second solution of Equation (4.1) must satisfy Equation (4.1)

The DE: $$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0$$

The assumed second solution: $u_2(x) = V(x)\, e^{mx}$

We must have: $$\frac{d^2\left[V(x)e^{mx}\right]}{dx^2} + a\,\frac{d\left[V(x)e^{mx}\right]}{dx} + b\left[V(x)e^{mx}\right] = 0 \qquad \text{(c)}$$

One would find: $$\frac{d\left[V(x)e^{mx}\right]}{dx} = mV(x)e^{mx} + e^{mx}\frac{dV(x)}{dx}$$

and $$\frac{d^2\left[V(x)e^{mx}\right]}{dx^2} = m\left[mV(x)e^{mx} + e^{mx}\frac{dV(x)}{dx}\right] + me^{mx}\frac{dV(x)}{dx} + e^{mx}\frac{d^2V(x)}{dx^2}$$

After substituting the above expressions into Equation (c), we get:

$$\frac{d^2V(x)}{dx^2} + (2m+a)\frac{dV(x)}{dx} + \left(m^2 + am + b\right)V(x) = 0 \qquad (4.10)$$

Since $m^2 + am + b = 0$ in Equation (4.3), and $m = m_1 = m_2 = a/2$ in Equation (b), both 2nd and 3rd term in Equation (4.10) drop out. We thus only have the first term to consider in the following special form of 2nd order DE:

$$\frac{d^2V(x)}{dx^2} = 0 \qquad (4.11)$$

The solution of DE in Equation (4.11) is: **V(x) = x**

The solution V(x) = x leads to the second solution of the DE

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0$$

in Case 3 with $a^2 - 4b = 0$ as:

$$u_2(x) = V(x)e^{mx} = xe^{mx} = xe^{-\frac{ax}{2}}$$

The complete solution of the DE in this case thus becomes:

$$u(x) = u_1(x) + u_2(x)$$

or

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x)e^{-\frac{ax}{2}} \qquad (4.12)$$

Summary on Solutions of 2nd Order Homogeneous DEs

The equation:

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (4.1)$$

with TWO given conditions

The solutions

Case 1: $a^2 - 4b > 0$:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2-4b}\; x/2} + c_2 e^{-\sqrt{a^2-4b}\; x/2}\right) \qquad (4.6)$$

Case 2: $a^2 - 4b < 0$:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b-a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b-a^2}\right)x\right] \qquad (4.8)$$

Case 3: $a^2 - 4b = 0$: — A special case

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2\, x e^{-\frac{ax}{2}} = (c_1 + c_2 x)e^{-\frac{ax}{2}} \qquad (4.12)$$

where c_1, c_2, A and B are arbitrary constants to be determined by given conditions

Example 4.1 Solve the following differential equation (p.84):

$$\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0 \quad \text{(a)}$$

Solution:

We have a = 5 and b = 6, by comparing Equation (a) with the typical DE in Equation (4.1). This will lead to:

$a^2 - 4b = 5^2 - 4x6 = 25 - 24 = 1 > 0$ - a Case 1 situation with $\sqrt{a^2 - 4b} = \sqrt{1} = 1$

Consequently, we may use the standard solution in Equation (4.6) to be:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\ x/2} + c_2 e^{-\sqrt{a^2 - 4b}\ x/2}\right)$$

or

$$u(x) = e^{-5x/2}\left(c_1 e^{x/2} + c_2 e^{-x/2}\right) = c_1 e^{-2x} + c_2 e^{-3x}$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions

Example 4.2 solve the following equation with given conditions (p. 84):

$$\frac{d^2u(x)}{dx^2}+6\frac{du(x)}{dx}+9u(x)=0 \qquad \text{(a)}$$

with given conditions: u(0) = 2 (b)

and
$$\left.\frac{du(x)}{dx}\right|_{x=0}=0 \qquad \text{(c)}$$

Solution:

Again by comparing Equation (a) with the typical DE in Equation (4.1), we have: a = 6 and b = 9. Further examining $a^2 - 4b = 6^2 - 4x9 = 36 - 36 = 0$, leading to special Case 3 in Equation (4.12) for the solution:

$$u(x)=c_1e^{-\frac{ax}{2}}+c_2xe^{-\frac{ax}{2}}=(c_1+c_2x)e^{-\frac{ax}{2}} \qquad (4.12)$$

or
$$u(x)=(c_1+c_2x)e^{-\frac{6}{2}x}=(c_1+c_2x)e^{-3x} \qquad \text{(d)}$$

Use Equation (b) for Equation (d) will yield $c_1 = 2$, leading to: $u(x)=(2+c_2x)e^{-3x}$ (e)

Differentiating Equation (e) with condition in Equation © will lead to the following manipulation:

$$\left.\frac{du(x)}{dx}\right|_{x=0}=\left[e^{-3x}(c_2)-3e^{-3x}(2+c_2x)\right]_{x=0}=c_2-6=0$$

So, we solve for $c_2 = 6$

Hence the complete solution of Equation (a) is: $u(x)=2(1+3x)e^{-3x}$

Part 2

Application of 2nd Order Homogeneous DEs for Mechanical Vibration Analysis

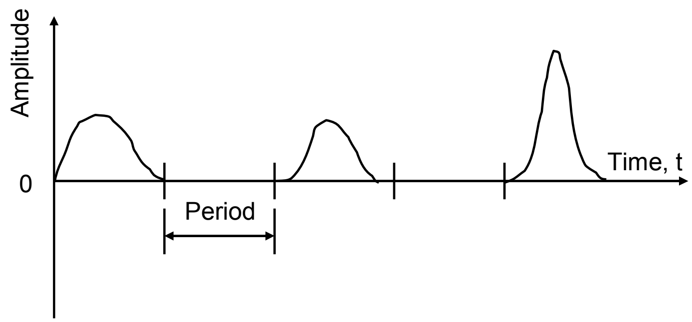
Mechanical vibration is a form of oscillatory motion of a solid or solid structure of a machine.

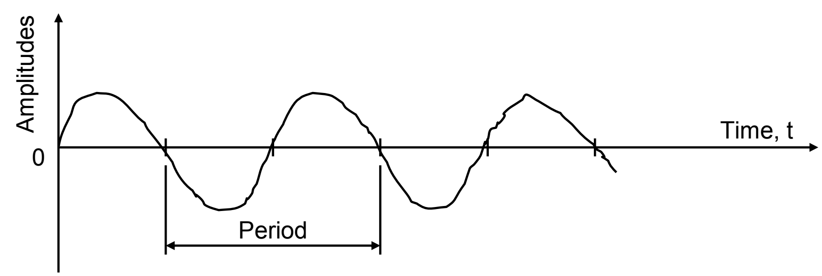
Common Sources of Mechanical Vibrations:

(1) Time-varying Mechanical force or pressure.
(2) Fluid induced vibration (e.g. intermittent wind, tidal waves, etc.)
(3) Acoustics and ultrasonic.
(4) Random movements of supports, e.g. seismic
(5) Thermal, magnetic, etc.

Common types of Mechanical Vibrations:

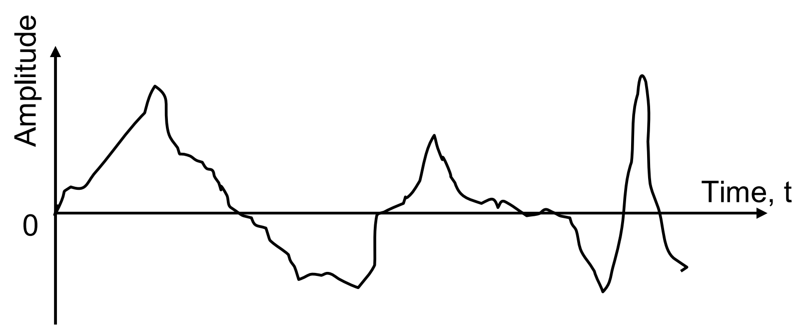
(1) With constant amplitudes and frequencies:





(2) With variable amplitudes but constant frequencies

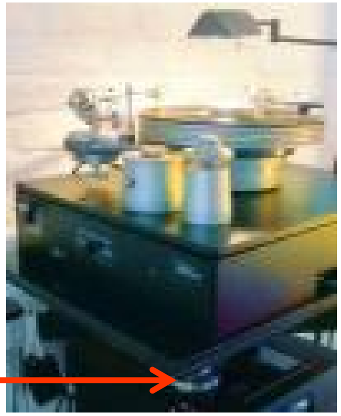
(3) With random amplitudes and frequencies:



Mechanical vibrations, in the design of mechanical systems, is normally undesirable occurrence, and engineers would attempt to either reduce it to the minimum appearance, or eliminate it completely.

"Vibration Isolators" are commonly designed and used to minimize vibration of mechanical systems, such as:

Benches for high-precision instruments



Vibration isolators

Suspension of heavy-duty truck

Vibration isolators

Design of vibration isolators requires analyses to quantify the amplitudes and periods of the vibratory motion of the mechanical system – a process called "mechanical vibration analysis"

The three types of mechanical vibration analyses by mechanical engineers:

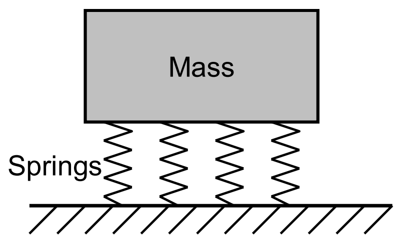
Mechanical vibration requires: Mass, spring force (elasticity), damping factor and initiator

A. Free vibration analysis:

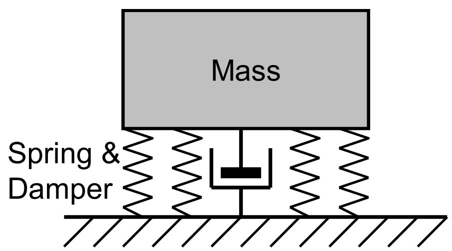
The mechanical system (or a machine) is set to vibrate from its initial equilibrium condition by an initial instantaneous disturbance (either in the form of a force or a displacement). this disturbance does not exist while the machine is vibrating.

There are two types of free vibrations:

- Simple mass-spring system:

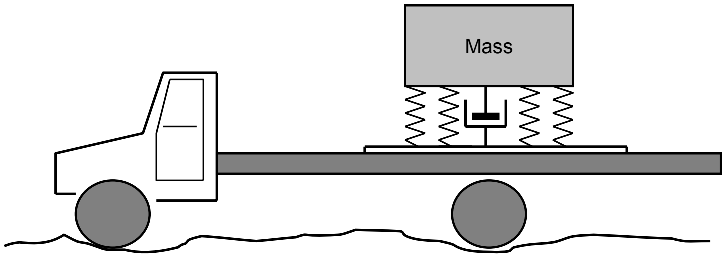


- Damped vibration system:



B. Forced vibration analysis:

Vibration of the mechanical system is induced by cyclic loading at all times.

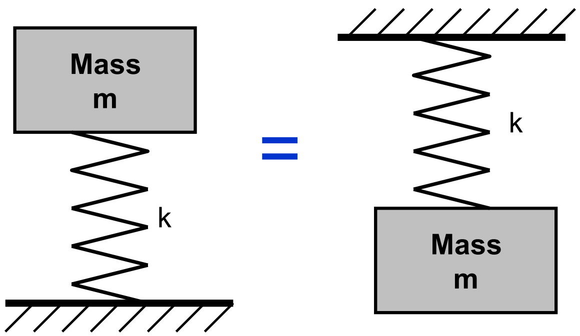


C. Modal analysis

To identify natural frequencies of a solid machine at various modes of vibration

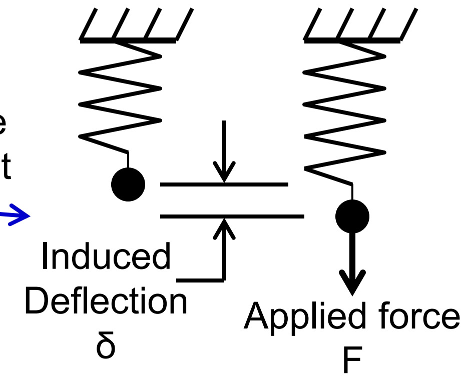
Physical Modeling of Mechanical Vibrations

The simplest model for mechanical vibration analysis is a MASS-SPRING system:



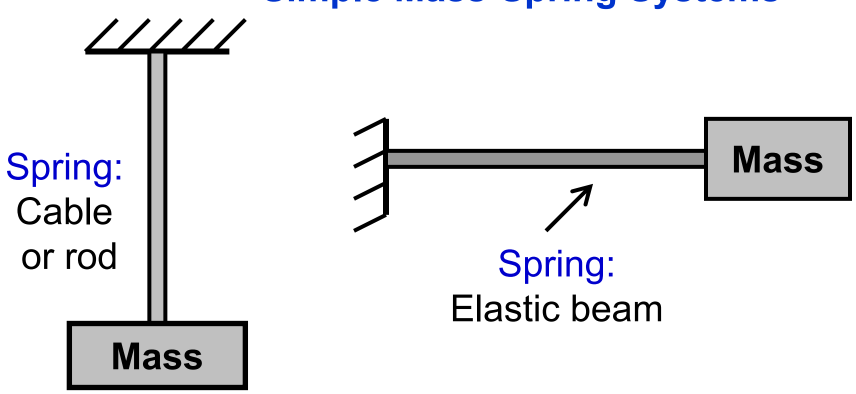
with m = mass, and
k = spring constant

- k is defined as the amount of force required to deflect a certain amount of the spring = F/δ
- So, k has a unit of lb_f/in or N/m
- k is a property of a given spring

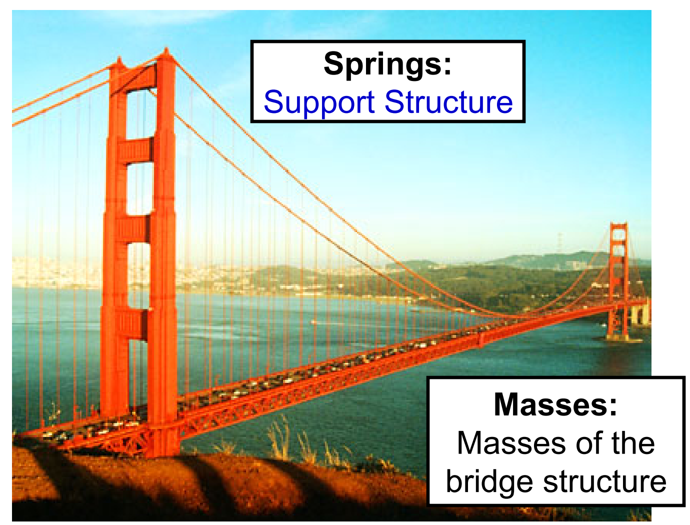


- The spring in this system is to support the mass
- Springs in the system need not to be "coil" springs
- Any ELASTIC solid support can be viewed as a "spring"

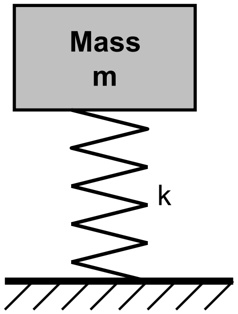
Simple Mass-Spring Systems

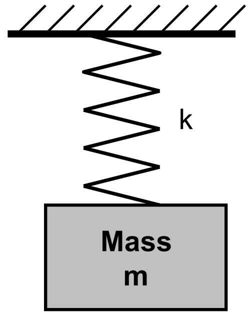


Complex System



Simple Mass-Spring Systems in Free Vibration

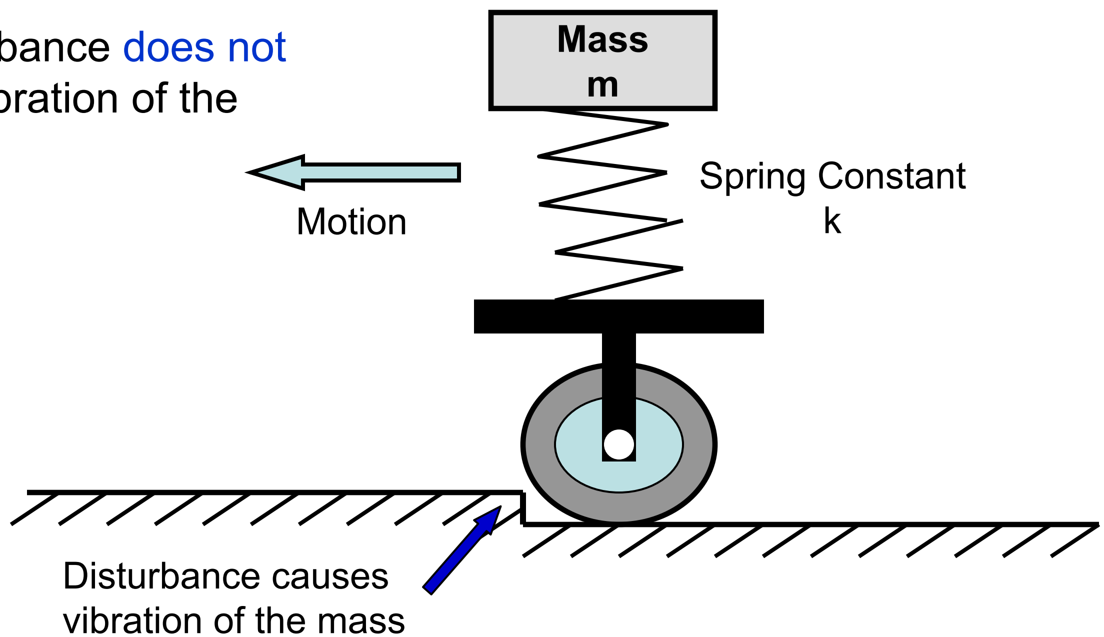




It takes a **MASS** and **SPRING** (or elastic) support to get the vibration of the mass going

The physical phenomena of solids in free vibration is that the vibration of solid is produced by an instantaneous disturbance either in the form of a force or deformation of the supporting spring.

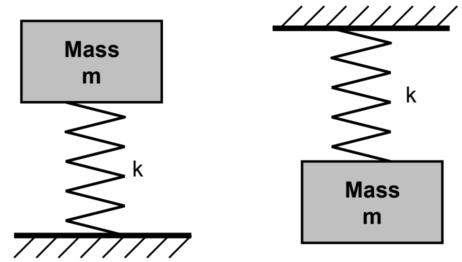
This initial disturbance does not exist after the vibration of the solids



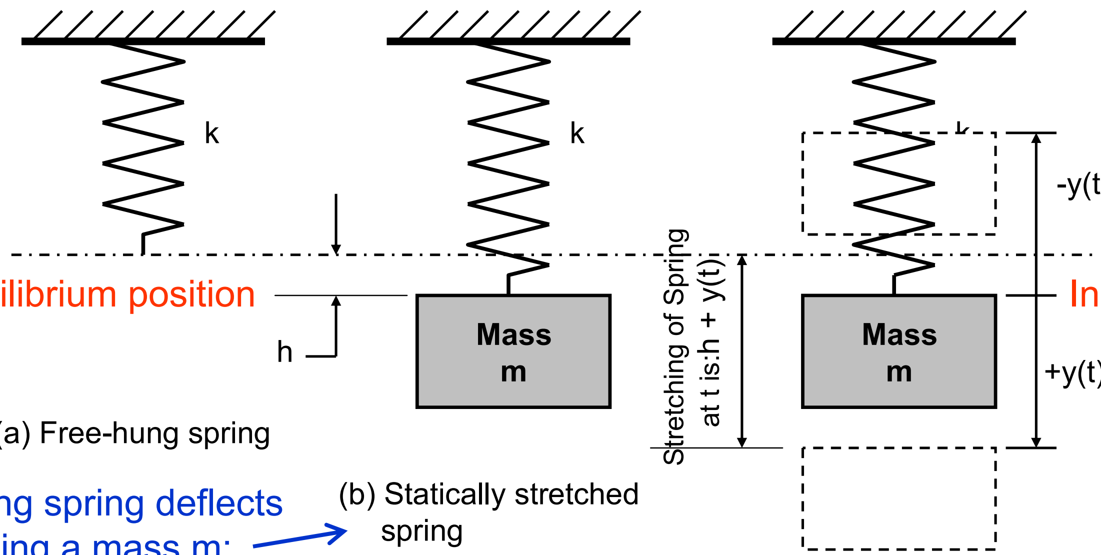
Mathematical Formulation of Simple Mass-Spring Systems in Free Vibration

As we said it before: "It takes a **MASS** and a **SPRING** (or elastic) support to get the vibration of the mass going.

So, the simplest physical model for a mechanical vibration system is like what is shown



Physical Model for Mathematical Formulation



Initial equilibrium position

Initial equilibrium position at time zero

●Sign convention: +ve downward

(1) Begins with: (a) Free-hung spring

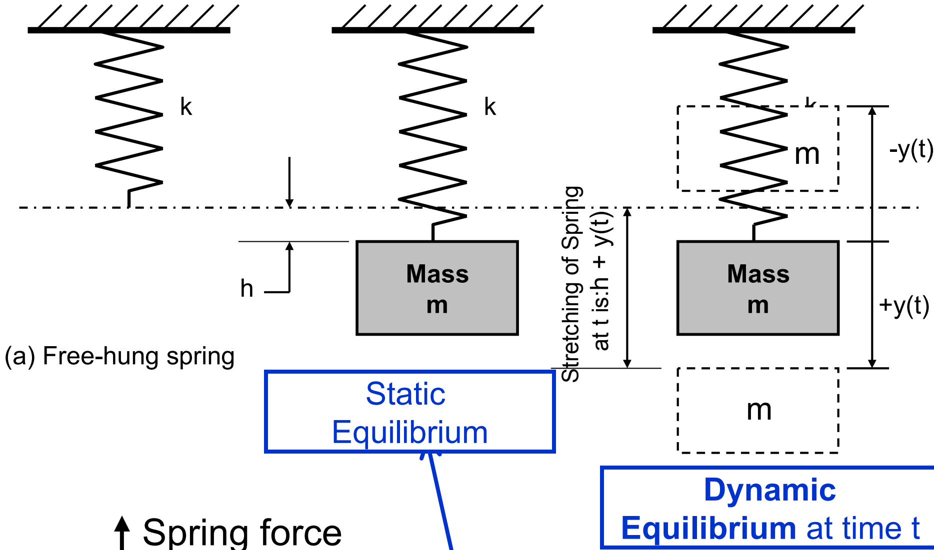
(2) The free-hung spring deflects upon attaching a mass m: (b) Statically stretched spring

(c) A vibrating mass at time t

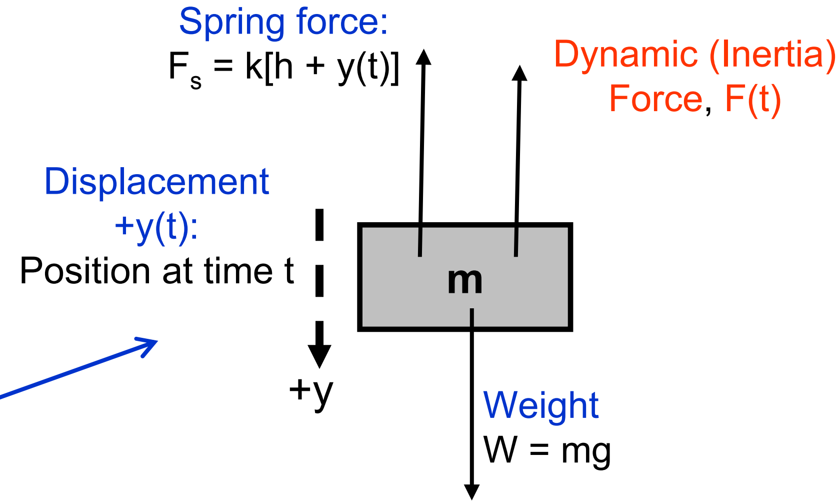
(3) A small instantaneous "push-down" is applied to the mass and release quickly.
We can expect the mass to bounce down and up passing its initial equilibrium position.

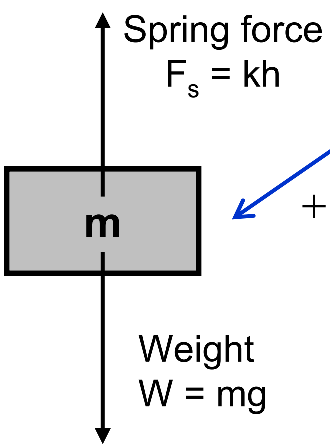
Mathematical Formulation of Mass-Spring System

with **no** air resistance to the motion



Forces: Weight (**W**); Spring force (**F_s**)
Dynamic force, **F(t)**





$$+\downarrow \sum F_y = W - F_s = 0$$

$$\therefore \quad mg = kh$$

Equilibrium of forces acting on the mass at given time t satisfies Newton's 1st Law:

$$+\downarrow \sum \left[-F(t) - F_s + W\right] = 0$$

But since we have the dynamic force to be: $F(t) = m\dfrac{d^2y(t)}{dt^2}$

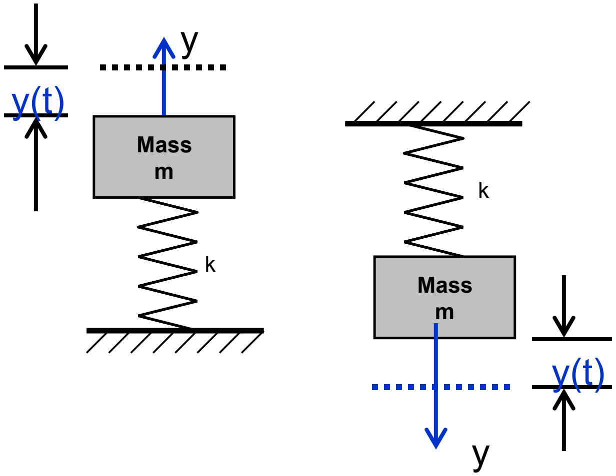
and the spring force to be F_s = k[h + y(t)], we should have:

$$-m\frac{d^2y(t)}{dt^2} - k\left[h + y(t)\right] + mg = 0$$

But mg = kh from the static equilibrium condition, after substituting it into the above equation, we have the following 2nd order differential equation for the instantaneous position y(t) for the vibrating mass:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (4.14)$$

Solution of differential equation (4.14) for simple mass-spring vibration



$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (4.14)$$

where y(t) = instantaneous position of the mass

Re-writing the equation in the form:

$$\frac{d^2y(t)}{dt^2} + \frac{k}{m}y(t) = 0 \qquad (4.14a)$$

The solution of Equation (4.14) can be obtained by comparing Equation (4.14a) with the typical 2nd order DE in Equation (4.1):

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (4.1)$$

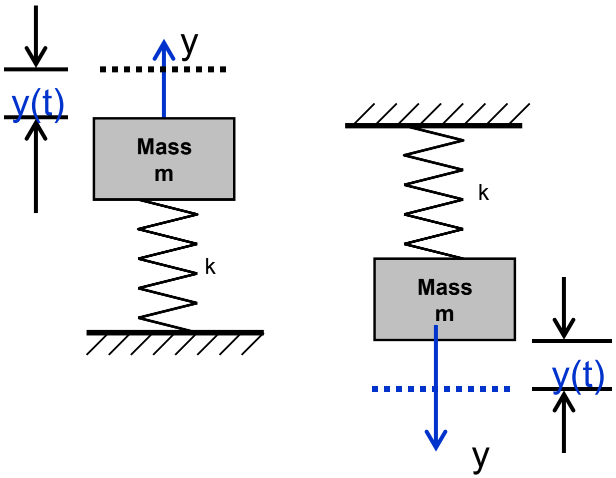
We may find that a = 0 and b = k/m after the comparison. The solution of Equation depends on the discriminator: $a^2 - 4b$. Since k = spring constant-a property of the spring and m = mass of the vibrating solid, the equivalent coefficient b is a +ve real number. Consequently, we have:

$a^2 - 4b = 0 - 4(k/m) < 0$, which is a Case 2 for the solution, as shown in Equation(4.8)

$$y(t) = A\,Cos\sqrt{\frac{k}{m}}\,t + B\,Sin\sqrt{\frac{k}{m}}\,t \qquad (4.15)$$

where A, B are arbitrary constants to be determined by given conditions

Solution of differential equation (4.14) for simple mass-spring vibration



$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (4.14)$$

where y(t) = instantaneous position of the mass

Re-writing the equation in the form:

$$\frac{d^2y(t)}{dt^2} + \frac{k}{m}y(t) = 0 \qquad (4.14a)$$

The common expression for the solution of Equation (4.14) is:

$$y(t) = c_1\,Cos\,\omega_o\,t + c_2\,Sin\,\omega_o\,t \qquad (4.16)$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions, and

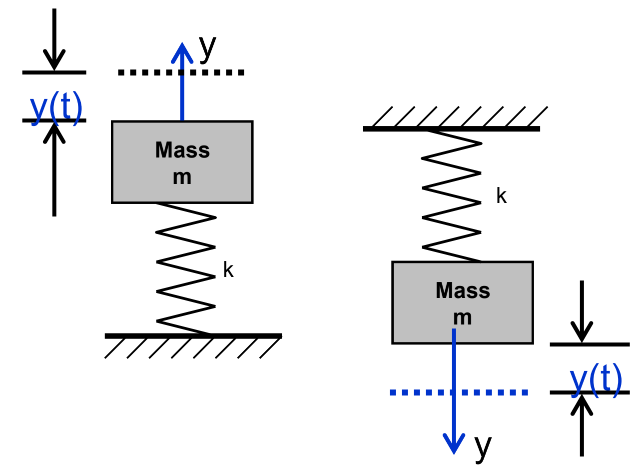
$$\omega_o = \sqrt{\frac{k}{m}} \qquad (4.16a)$$

The ω_o in Equation (4.16a) is called the "*circular, or angular frequency*" of the mass-spring vibration system. Often its represents the "***natural frequency***" of the system. The unit is Rad/s.

Corresponding to the angular frequency ω_o is the real frequency of the vibration:

$$f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (4.17)$$

Graphical representation of free-vibration of mass-spring systems

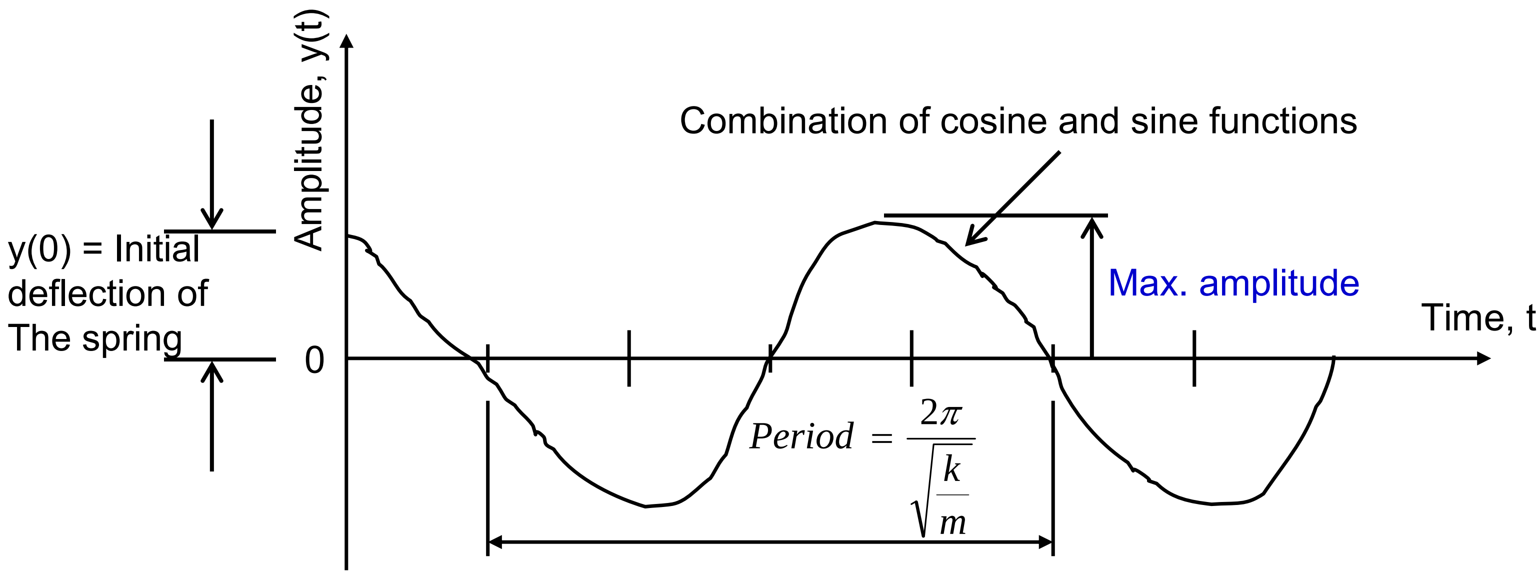


Mathematical solution:

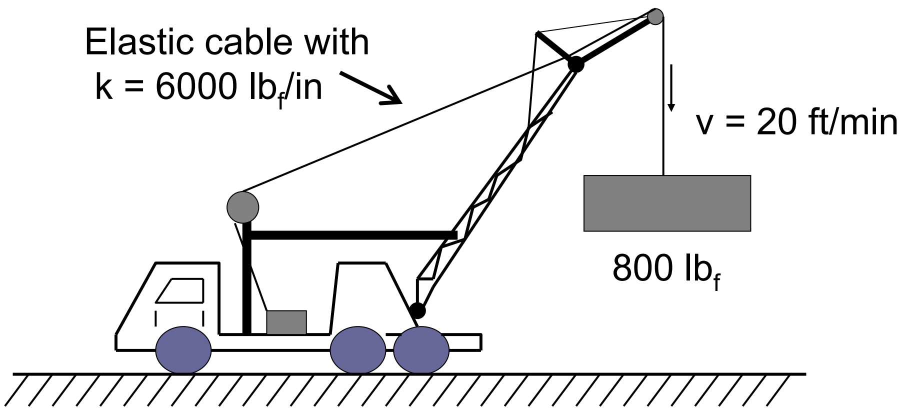
$$y(t) = c_1 \; Cos\,\omega_o\, t + c_2 \; Sin\,\omega_o\; t \qquad (4.16)$$

y(t) consists of cosine and sine functions of variable t (the time)

So, it is an oscillatory function, oscillating about the "zero-time" axis, with amplitudes of vibration y(t):



Example 4.3 An unexpected case for engineers to consider in their design and operation of an unloading process.



Description of the problem:
A truck is unloading a heavy machine weighing 800 lb_f by a crane.
The cable was suddenly seized (jammed) at time t from a descending velocity of v = 20 ft/min

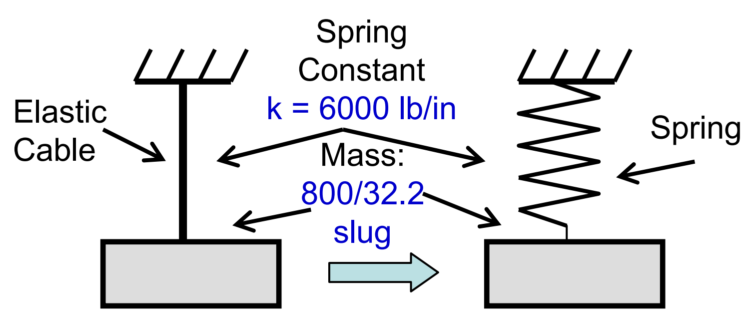
One may expect the heavy machine will undergo an "up-down-up" vibration after such seizure.

Determine the following:

(A) The frequency of vibration of the machine that is seized from descending
(B) The maximum tension in the cable induced by the vibrating machine, and
(C) The maximum stress in the cable if the stranded steel cable is 0.5 inch in diameter
(D) Would the cable break if its maximum allowable strength is 40,000 psi?

Solution:

Because the machine is attached to an elastic cable, which has the characteristics of a "spring," we may simulate this situation to a simple mass-spring systems:



The frequency and amplitudes of the vibrating machine can thus be evaluated by the expressions derived for the simple mass-spring system.

(a) **The frequency of vibration** of the machine is given in Equation (4.16) and (4.17).

Numerically, we have the following:

The circular frequency is:

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000\,x12}{800/32.2}} = 53.83\ rad/s$$

which leads to the frequency to be:

$$f = \frac{\omega_o}{2\pi} = 8.57\ \ cycles/s$$

(b) The **maximum tension in the cable**:

The maximum tension in the cable is determined with the maximum total elongation of the steel cable, i.e. the maximum amplitude of the vibration of the machine after the cable is seized.
To get the amplitude of the vibrating machine, we need to solve a differential equation that has the form as shown in Equation (4.14) and the appropriate conditions.

The following formulation is obtained:

The DE:
$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (4.14)$$

The conditions: y(0) = 0 and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 20\ ft/\min = 0.3333\ ft/s$ (a)

Initial velocity (velocity at the time of seizure)

The solution of Equation (4.14) is: $$y(t) = c_1 \, Cos\,\omega_o\, t + c_2 \, Sin\,\omega_o\, t \qquad (4.16)$$

or $$y(t) = c_1 \, Cos\, 53.83\, t + c_2 \, Sin\, 53.83 t \qquad (b)$$

with ω_o = 53.83 Rad/s as computed in Part (a) of the solution.

The arbitrary constants c_1 and c_2 in Equation (b) can be determined by using the conditions given in Equation (a), with c_1 = 0 and c_2 = 0.0062.

We thus have the amplitude of the vibrating machine in the following form:

$$y(t) = 0.0062\, Sin\, 53.83 t \qquad (c)$$

From which, we obtain the maximum amplitude from Equation (c) to be:

y_{max} = 0.0062 ft (d)

The corresponding maximum tension in the cable is:

$$T_m = k\, y_{\max} + W = (6000\, x 12)\, x 0.0062 + 800 = 1246\ lb_f$$

(C) The **maximum stress in the cable** is obtained by the following expression:

$$\sigma_{\max} = \frac{T_m}{A} = \frac{1246}{\frac{\pi (0.5)^2}{4}} = 6346\ \ psi \qquad (e)$$

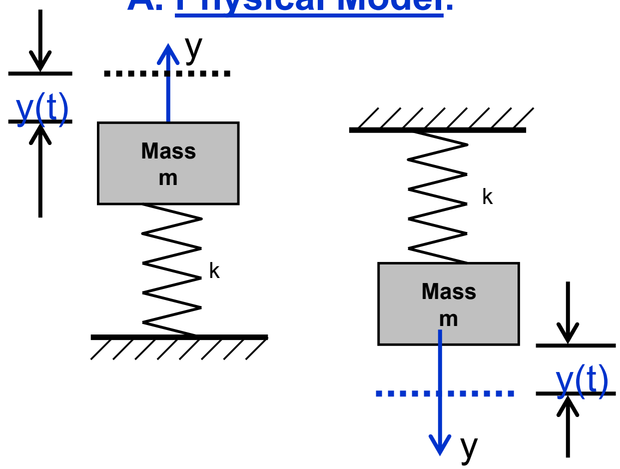
(d) **Interpretation of the analytical result**:

The cable will not break, because the maximum induced stress σ_{max} = 6346 psi << σ_a where σ_a is the maximum strength of the cable material = 40,000 psi

Simple Mass-Spring Systems in Free Vibration

In the cases of simple Mass-Spring systems:

A. Physical Model:



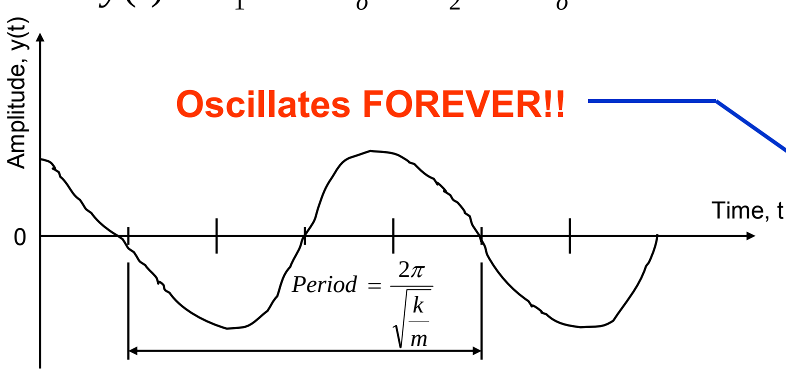
B. Mathematical Model:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0$$

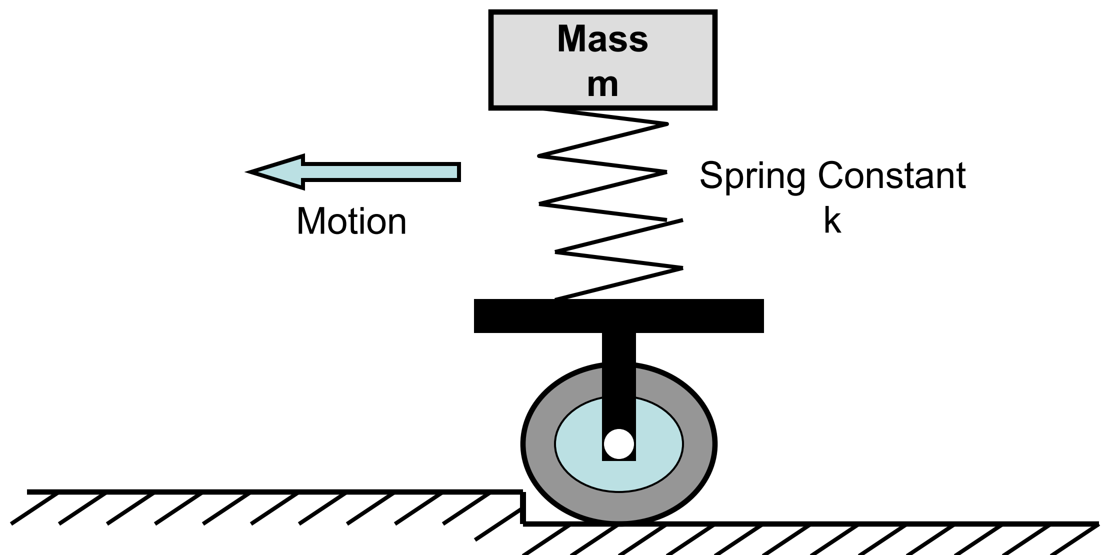
C. Analytical Solution:

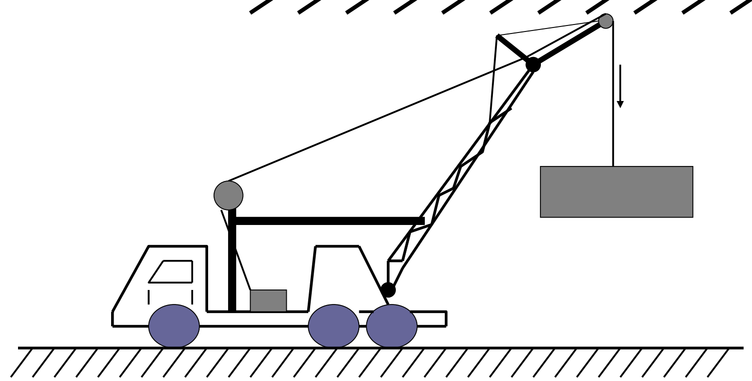
$$y(t) = c_1\,Cos\,\omega_o\,t + c_2\,Sin\,\omega_o\,t$$

Oscillates FOREVER!!



D. Examples of Applications:





E. Interpretation of Results:

NOT realistic – the mass CANNOT oscillate forever!! Vibration will eventually stop.

Simple Damped Mass-Spring Systems in Free Vibration

Question: What makes free-vibration of a mass-spring system to stop after time t in reality??

Answer: It is the "**damping effect**" that makes the free vibration of mass-spring system to stop after time t

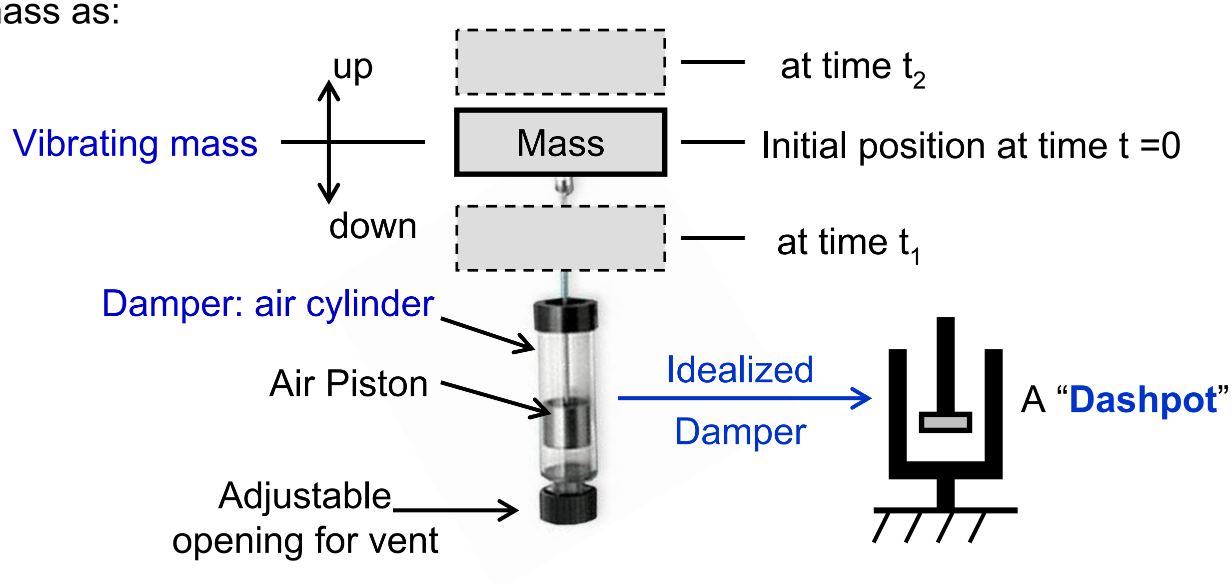
So, "Damped" free vibration of solids is a more realistic phenomenon

Sources of Damping in Mechanical Vibrations

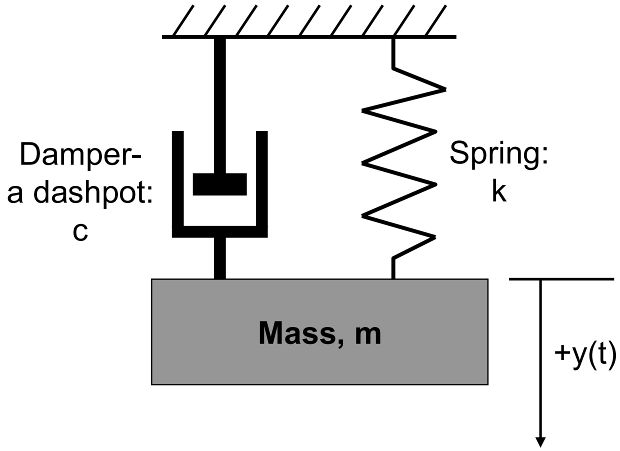
- Resistance by the air surrounding the vibrating mass – easier to model
- Internal friction of the spring during deformations – a material science topic

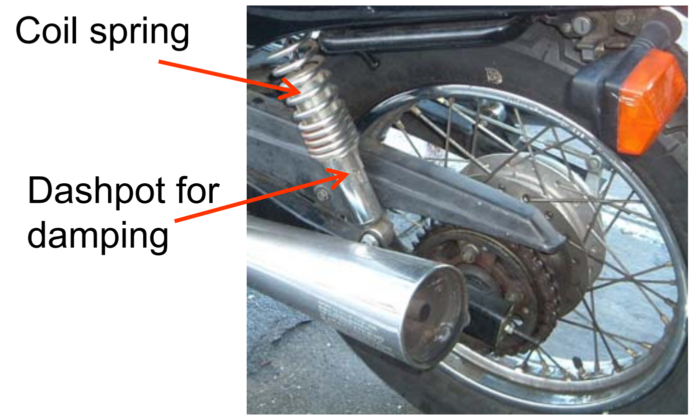
Physical model of Damped Mass-Spring Systems in Free Vibration

Because damping of a simple mass-spring vibration system in induced by air resistance to the moving mass, we can use an "air cylinder with adjustable air vent to regulate the air resistance to the moving mass as:



Physical modeling of Damped Mass-Spring Systems in Free Vibration





Real-world Application with "Coilovers" in motorcycle suspension

The damper in the physical model is characterized by a damping coefficient c – similar to the situation of a spring characterized by spring constant k.

The damping coefficient c is specified by manufacturer of the damper (a dashpot)

Because the corresponding damping force is related to the air resistance to the movement of the mass, and the resistance R is proportional to the velocity of the moving mass. Mathematically, we have:

$$R(t) \propto Velocity\ of\ moving\ mass\ \left(= \frac{dy(t)}{dt}\right)$$

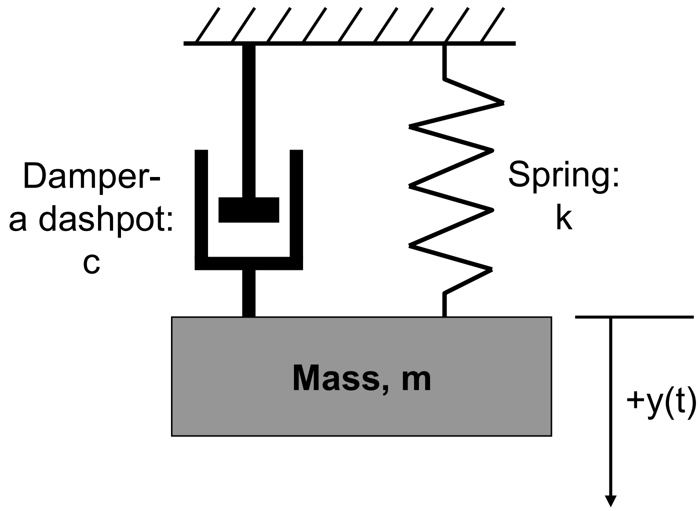
where y(t) is the distance the mass has traveled from its initial equilibrium position

Consequently, the damping force R(t) has the form:

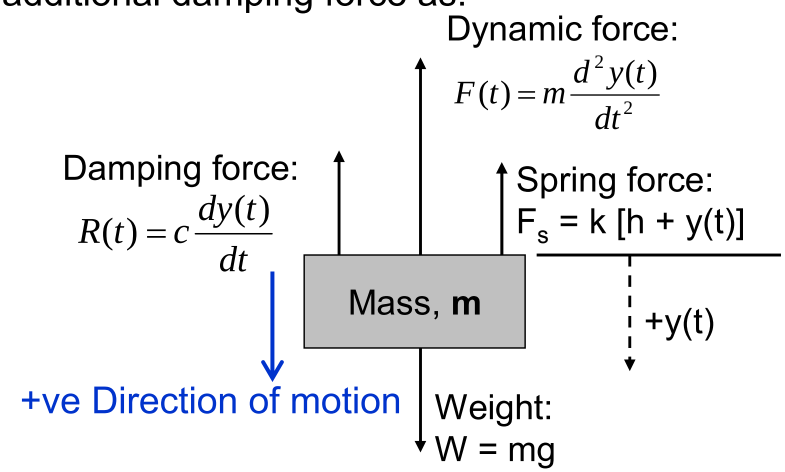
$$R(t) = c\frac{dy(t)}{dt} \tag{4.19}$$

in which c = damping coefficient

Mathematical modeling of Damped Mass-Spring Systems in Free Vibration



The mathematical expression of this physical model can be obtained by following similar procedure for the simple mass-spring system, with the inclusion of the additional damping force as:



By Newton's 1st Law in dynamic equilibrium:

$$+\downarrow \sum F_y = -F(t) - R(t) - F_s + W = 0$$

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) + kh - mg = 0$$

or

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) = 0 \qquad (4.20)$$

Equation (4.20) is a 2nd order homogeneous differential equation for the instantaneous position of the vibrating mass

Solution of Eq. (4.20) for Damped Mass-Spring Systems in Free Vibration

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) = 0 \tag{4.20}$$

If we re-write the equation in a different form:

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = 0 \tag{4.20a}$$

Now, if we compare Equation (4.20a) and the typical 2nd order homogeneous DE in Equation (4.1):

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \tag{4.1}$$

we will have a = c/m and b = k/m

We may obtain the solutions of Equation (4.20) depends on the signs of the discriminators (a^2 – 4b) or $(c/m)^2$ – 4(k/m) > 0, or =0, or <0. effectively, we will look for the 3 possible cases:

Case 1: $(c/m)^2 - 4(k/m) > 0$, or $c^2 - 4mk > 0$

Case 2: $(c/m)^2 - 4(k/m) = 0$ or $c^2 - 4mk = 0$

Case 3: $(c/m)^2 - 4(k/m) < 0$ or $c^2 - 4mk < 0$

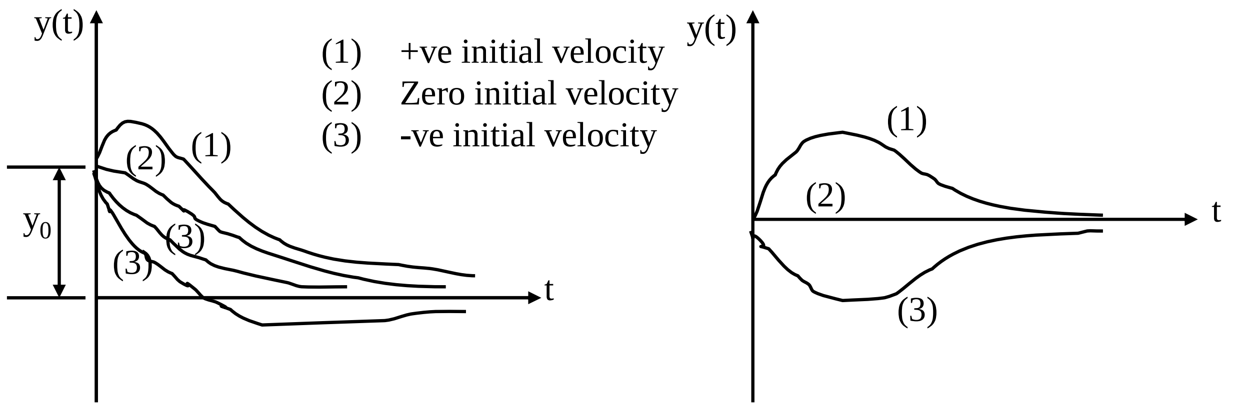
<u>Case 1</u>: **c^2 - 4mk > 0** (Over-damping situation):

The solution in Equation (4.6) is applied:

$$y(t) = e^{-(c/2m)t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right) \tag{4.22}$$

where $\Omega = \sqrt{c^2 - 4mk}\,/\,(2m)$ and A, B are arbitrary constants to be determined by two given conditions

Graphical representation of the instantaneous position of the vibrating mass are:



(a) With +ve initial displacement, y_0 (b) With negligible initial displacement

Observations:

- There is **no oscillatory motion** of the mass.
- There can be an initial increase in the displacement, followed by continuous decays in the amplitudes of vibration
- The amplitudes of vibration usually decays quickly in time

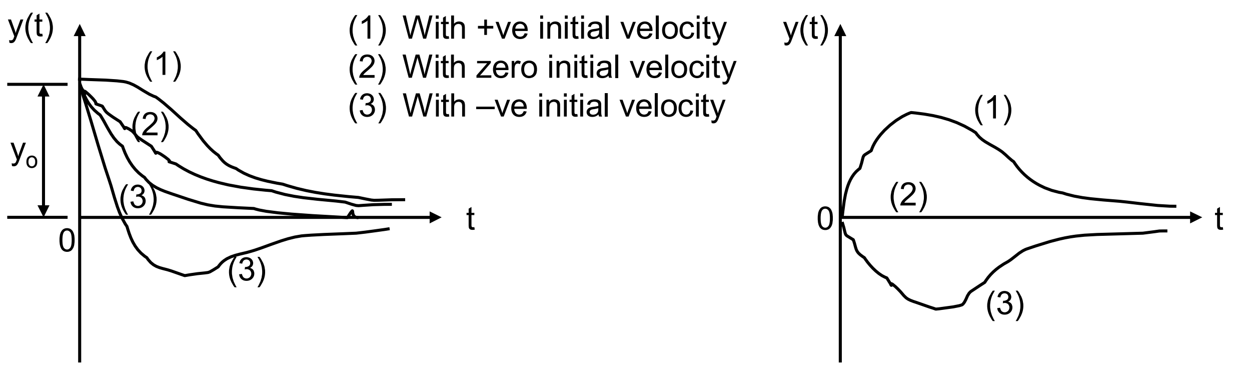
- A desirable situation in abating (mitigating) mechanical vibration

Case 2: $c^2 - 4mk = 0$ (Critical damping):

Solution of Equation (4.20) is in the form of Equation (4.12):

$$y(t) = e^{-\left(\frac{c}{2m}\right)t}\left(A + Bt\right) \tag{4.23}$$

Graphical representation of Equation (4.23) is:



(a) With +ve initial displacement

(b) With negligible initial displacement

Observations:

- There is no oscillatory motion of the mass by theory
- Amplitudes reduce with time, but take longer to "die down" than in the case of "over-damping"
- May become an unstable situation of vibration

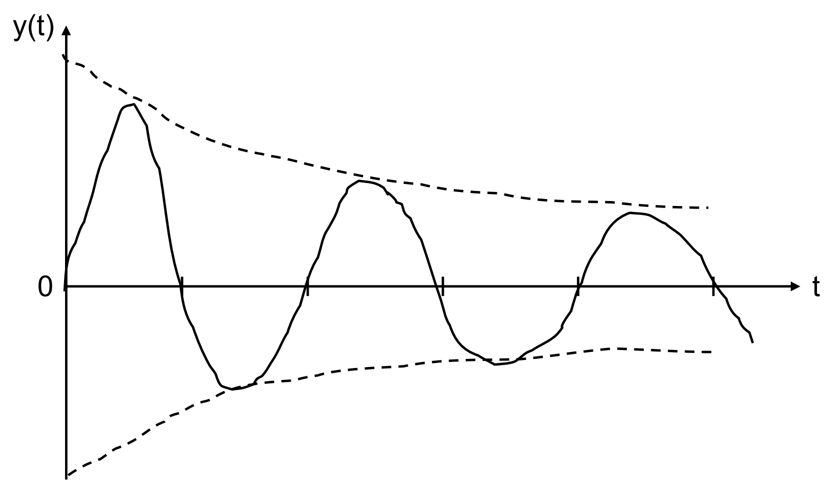
Case 3: **$c^2 - 4mk < 0$** ((Under damping):

Solution of Equation (4.20) in this case is expressed in Equation (4.8)

$$y(t) = e^{-\left(\frac{c}{2m}\right)t}\left(A\,Cos\,\Omega t + B\,Sin\,\Omega t\right) \tag{4.24}$$

where $\Omega = \sqrt{4mk - c^2}\;/\;(2m)$ and A, B are arbitrary constants

Graphical representation of Equation (4.24) is:



Observations:

- The only case of damped vibration that has oscillatory motion of the mass
- The amplitudes of each oscillatory motion of the mass reduces continuously but they take a long time to “die down”
- “Under damping” is thus the least desirable situation in machine design

Part 3

Review solution method of second order, non-homogeneous ordinary differential equations

- Applications in forced vibration analysis
- Resonant vibration analysis
- Near resonant vibration analysis

Typical second order, non-homogeneous ordinary differential equations:

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = g(x) \qquad (4.25)$$

Non-homogeneous term

Solution of Equation (4.25) consists **TWO** components:

$\boxed{\text{Solution } u(x)}$ **=** $\boxed{\text{Complementary solution } u_h(x)}$ **+** $\boxed{\text{Particular solution } u_p(x)}$

$$u(x) = u_h(x) + u_p(x) \qquad (4.26)$$

The complementary solution $u_h(x)$ is the solution of the homogeneous part of Equation (4.25), i.e.:

$$\frac{d^2u_h(x)}{dx^2} + a\frac{du_h(x)}{dx} + bu_h(x) = 0 \qquad (4.27)$$

Equation (4.27) is similar to the typical 2nd order homogeneous differential equation in Equation (4.1).

Solutions are available in Equation (4.6) for Case 1 with $a^2 - 4b > 0$; Equation (4.7) for Case 2 with $a^2 - 4b < 0$; and Equation (4.12) for Case 3 with $a^2 - 4b = 0$

Determination of particular solution $u_p(x)$

There is **NO** fixed rule for deriving $u_p(x)$

However, the guideline that one may use is by ASSUMING a function that is **SIMILAR** to the non-homogeneous part of the DE, e.g., g(x) in Equation (4.25):

g(x) with specified coefficients	**$u_p(x)$** with unknown coefficients need to be determined
Polynomial of order n: $g(x) = ax^4 + b\,x^2 + cx + d$ (order 4)	Polynomial of order n: $u_p(x) = A_o + A_1x + A_2x^2 + A_3x^3 + A_4x^4$ (order 4)
Trigonometric functions: $g(x) = a\,\text{sine}(\alpha x)$, or $b\cos(\alpha x)$, or $g(x) = a\cos(\alpha x) + b\sin(\alpha x)$	**ALL** trigonometric functions: up $(x) = A\cos(\alpha x) + b\sin(\alpha x)$
Exponential functions: $g(x) = ae^{bx}$	Exponential functions: $u_p(x) = Ae^{bx}$
Combination of functions: $g(x) = ax^3 + b\cos(\alpha x) + ce^{-dx}$	Combination of similar functions: $u_p(x) = (Ax^3+Bx^2+Cx+D)$ $+ [E\cos(\alpha x) + F\sin(\alpha x)] + Ge^{-dx}$

The coefficients in the assumed $u_p(x)$ are determined by comparing terms after its substituting into the DE in Equation (4.25)

The coefficients in assumed $u_p(x)$ are determined by comparing terms after its substituting into the DE in Equation (4.25):

$$\frac{d^2 u_p(x)}{dx^2} + a\frac{du_p(x)}{dx} + bu_p(x) = g(x) \qquad (4.28)$$

Your assumed $u_p(x)$

Self study: Examples (4.4), (4.5) and (4.7)

Example 4.6 Solve the following DE (P. 97):

$$\frac{d^2 y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = Sin2x \qquad (a)$$

Equation (a) is a non-homogeneous equation. So the solution by following Equation (4.26)is:

$$y(x) = y_h(x) + y_p(x) \qquad (b)$$

The complementary solution yh(x) in Equation (b) is obtained from homogeneous part of Equation (a) as:

$$\frac{d^2 y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0 \qquad (c)$$

The solution of Equation (c) is by Case 1 with a^2-4b>0, or

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} \qquad (d)$$

To determine the particular solution $y_p(x)$:

Because the non-homogeneous part of the DE, g(x) = sin 2x in Equation (a), so the assumed $y_p(x)$ should include BOTH sine and cosine functions:

$$y_p(x) = A \sin 2x + B \cos 2x \qquad \text{(e)}$$

which leads to:

$$\frac{dy_p(x)}{dt} = 2ACos2x - 2BSin2x \quad \text{and} \quad \frac{d^2y_p(x)}{dt^2} = -4ASin2x - 4BCos2x$$

Substituting $y_p(x)$ in Equation (e) and its derivatives into Equation (a):

$$\frac{d^2y_p(x)}{dx^2} - \frac{dy_p(x)}{dx} - 2y_p(x) = Sin2x$$

(- 4A Sin 2x – 4B Cos 2x) –
(2A Cos 2x – 2BSin 2x) –
-2(A Sin 2x + B Cos 2x) = Sin 2x

g(x)

After re-arranging terms, we get:

(- 6A + 2B) Sin 2x + (-6B – 2A) Cos 2x = Sin 2x

By comparing the coefficients of the terms on both sides of the above expression, we get:

6A = 2B = 1 and – 2A – 6B = 0, from which we solve for: A = - 3/20 and B = 1/20

The particular solution is thus: $y_p(x)$ = -3 Sin2x/20 + Cos 2x/20, which leads to the solution of the DE in Equation (a) to be:

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + \left(-\frac{3}{20} Sin\, 2x + \frac{1}{20} Cos\, 2x \right)$$

Special Case in Determining Particular Solution $u_p(x)$

This case involves at least one term in the complementary solution of the DE coincides with the term of the function in the non-homogeneous part of the DE, i.e. g(x)

Example of the special case – **Example 4.8** (P. 99):

$$\frac{d^2u(x)}{dx^2} + 4u(x) = 2\,Sin\,2x \qquad \text{(a)}$$

By the usual procedure, we will get the complementary solution first by solving:

$$\frac{d^2u_h(x)}{dx^2} + 4u_h(x) = 0 \qquad \text{(b)}$$

The solution is: $u_h(x) = c_1 \text{ Cos } 2x + c_2 \text{ Sin } 2x$ (c)

where c_1 and c_2 are arbitrary constants

We realize the 2nd term in the solution of $u_h(x)$ in Equation (c) is of the same form of g(x) = 2 sin2x in Equation (a). So, it is a special case. We will see from the following derivation of $u_p(x)$ by the "normal" way will lead us to NOWHERE as we will see form the following derivation!

Since the non-homogeneous part of the DE, g(x) = 2 sin2x – a trigonometric function, the "normal" way would having us assuming the particular solution in the form:

$$u_p(x) = A \text{ Cos } 2x + B \text{ Sin } 2x \qquad \text{(d)}$$

Substituting the $u_p(x)$ in Equation (d) into Equation (a) will lead to the following ambiguous equality:

$$(0)Cos\,2x + (0)Sin2x = 2\,Sin2x$$

In no way we can solve the coefficients A and B in Equation (d). Another way of assuming $u_p(x)$ is needed

Particular solution for special cases:

Let us modify the assumed $u_p(x)$ in Equation (d) for the special case:

$$u_p(x) = x\,(A \, Cos\, 2x + B\, Sin\, 2x) \qquad \text{(e)}$$

Now if we follow the usual procedure with the modified $u_p(x)$ in Equation (e) to DE in Equation (a), we need first to derive the following derivatives as:

$$\frac{du_p(x)}{dx} = A(-2xSin2x + Cos2x) + B(2xCos2x + Sin2x) \qquad \text{(f)}$$

and

$$\frac{d^2u_p(x)}{dx^2} = A[-4xCos2x - 2Sin2x - 2Sin2x]$$
$$= B[-4xSin2x + 2Cos2x + 2Cos2x] \qquad \text{(g)}$$

Upon substituting the above modified $u_p(x)$ in Equation (e) and the derivatives in Equations (f) and (g) into the DE in Equation (a), we will have:

(-4Ax Cos 2x – 2A Sin 2x – 2A Sin 2x – 4Bx Sin2x + 2B Cos 2x + 2B Cos 2x)
+ (4Ax Cos 2x + 4Bx Sin 2x) = 2 Sin 2x

from which we get: A = -1/2 and B = 0, which lead to: $u_p(x) = -\frac{x}{2} Cos2x$ (h)

The complete general solution of the DE in Equation (a) is the summation of $u_h(x)$ in Equation (c) and the $u_p(x)$ in Equation (h):

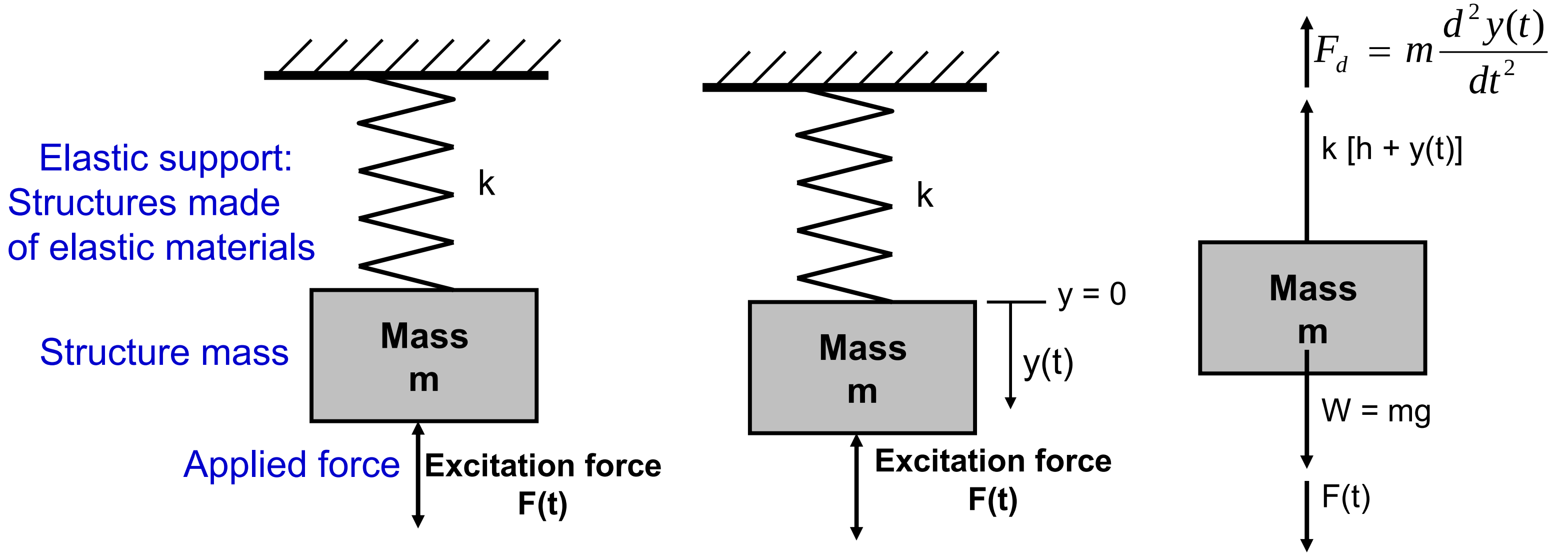
$$u(x) = u_h(x) + u_p(x) = c_1\, Cos\, 2x + c_2\, Sin\, 2x - \frac{x}{2} Cos\, 2x$$

Resonant Vibration Analysis

- This is one of several critical mechanical engineering (or structure) analyses
- Any machine or structure that is subjected to POTENTIAL CYCLIC (intermittent) loading is vulnerable to resonant vibration
- The consequence of resonant vibration is that the AMPLITUDES of the oscillatory motion of the structure continue to magnify in short time, resulting in overall structural failure

- Because resonant vibration of a machine or structure occur when it is subjected to CYCLIC loads, it is a "FORCED VIBRATION" case with forces acting to the vibrating solids at all times

The simplest physical model for forced vibration is a simple mass-spring system subjected to an exciting force F(t) where t = time:



The mathematical model for the above physical model can be derived by using Newton's First law:

$$+\uparrow \sum F_y = 0 \rightarrow -F_d - k[h + y(t)] + W + F(t) = 0$$

with $F_d = m\dfrac{d^2y(t)}{dt^2}$ from Newton's 2nd law

The differential equation for the instantaneous amplitudes of the vibrating mass under the influence of force F(t) becomes:

$$m\frac{d^2y(t)}{dt^2} + ky(t) = F(t) \qquad (4.31)$$

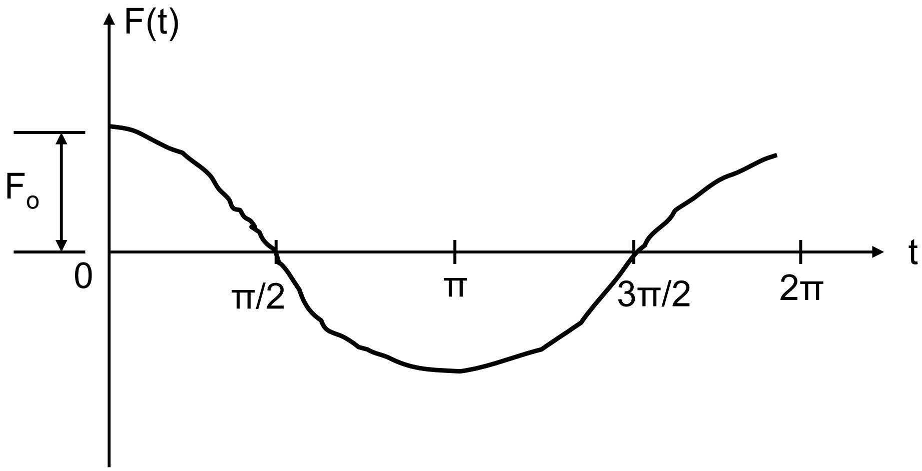
Forced Vibration of a Mass-Spring System subject to Cyclic Forces:

If we assume the applied force F(t) in Equation (4.31) is of cyclic nature following a cosine function, i.e.:

$$F(t) = F_o \, Cos\, \omega t \tag{4.32}$$

where F_o = maximum magnitude of the force, and ω is the circular frequency of the applied cyclic force

F(t) is graphically displayed as:



Upon substituting the expression of F(t) in Equation (4.32) into Equation (4.31), we have the governing differential equation for the amplitudes of the vibrating mass as:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = F_0\,Cos\,\omega t \tag{4.33}$$

Solution of Equation (4.33): $$m\frac{d^2y(t)}{dt^2}+k\,y(t)=F_0\,Cos\,\omega t \qquad (4.33)$$

or in a different form: $$\frac{d^2y(t)}{dt^2}+\frac{k}{m}y(t)=\frac{F_o}{m}Cos\,\omega t \qquad (4.33a)$$

or in yet another form: $$\frac{d^2y(t)}{dt^2}+\omega_o^2\,y(t)=\frac{F_o}{m}Cos\,\omega t \qquad (4.33b)$$

in which $\omega_0=\sqrt{\frac{k}{m}}$ is the circular frequency of the mass-spring system (a property of the mass-spring "structure")

Equation (4.33b) is a non-homogeneous 2nd order differential equation, and its solution is:

$$y(t) = y_h(t) + y_p(t)$$

The complementary solution $y_h(t)$ is obtained from the homogeneous part of the DE:

$$\frac{d^2y_h(t)}{dt^2}+\omega_0^2\,y_h(t)=0 \qquad (4.33)$$

Solution of Equation (4.33d) is:

$$y_h(t)=c_1\,Cos\,\omega_o t+c_2\,Sin\,\omega_o t \qquad (4.33e)$$

The particular solution of Equation (4.33b) can be assumed as:

$$y_p(t) = A \text{ Cos } \omega t + B \text{ Sin } \omega t \tag{4.34}$$

We will have: $\dfrac{dy_p(t)}{dt} = -A\omega Sin\omega t + B\omega Cos\,\omega t$ and $\dfrac{d^2y(t)}{dt^2} = -A\omega^2 Cos\,\omega t - B\omega^2 Sin\,\omega t$

Upon substituting the above into Equation (4.33b) with y(t) = y_p(t):

$$\frac{d^2y_p(t)}{dt^2} + \omega_0^2\, y_p(t) = \frac{F_0}{m} Cos\,\omega t$$

We have: $$\left(-A\omega^2 Cos\omega t - B\omega^2 Sin\omega t\right) + \omega_0^2\left(ACos\omega t + BSin\omega t\right) = \frac{F_0}{m} Cos\omega t$$

Upon comparing terms on both sides of the above equality:

$\left(-A\omega^2 + \omega_0^2 A\right) = \dfrac{F_0}{m}$ for the terms with Cosωt, leading to: $A = \dfrac{F_0}{m(-\omega^2 + \omega_0^2)}$

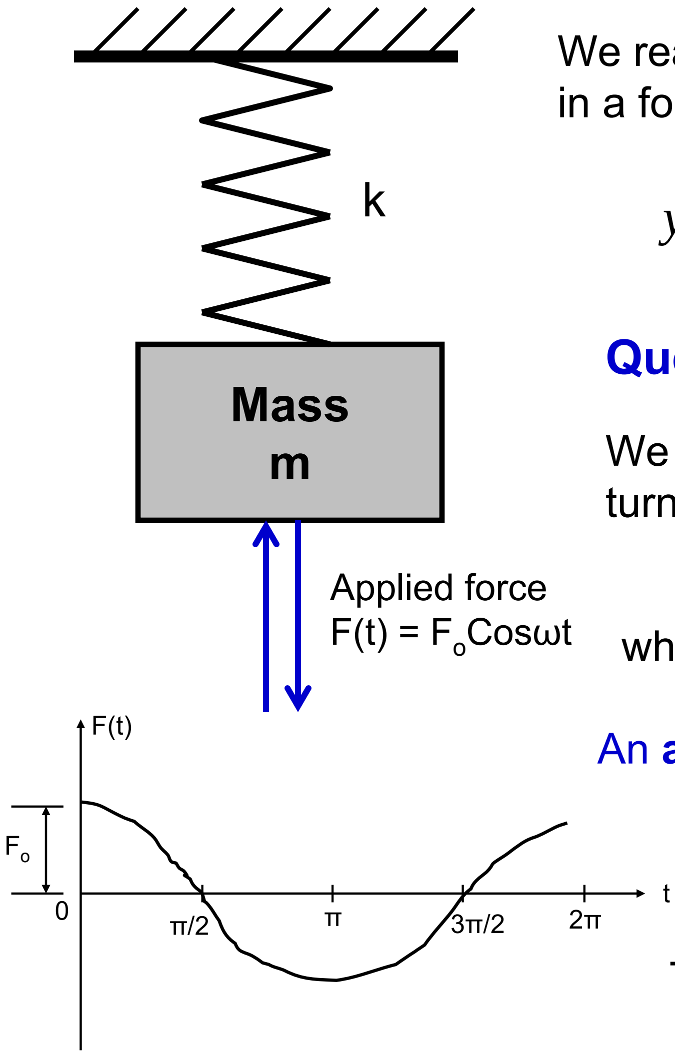
and for the term of Sinωt: $\left(-\omega^2 + \omega_0^2\right)B = 0$ leading to: B = 0

Thus, we have: $$y_p(t) = \frac{F_0}{m(-\omega^2 + \omega_o^2)} Cos\omega t$$

The complete solution of DE for forced vibration by cyclic force F(t) = F_o Cosωt in Equation (4.33) is:

$$y(t) = c_1\, Cos\omega_0 t + c_2\, Sin\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \tag{4.35}$$

with c_1 and c_2 to be the arbitrary constants determined by specified initial conditions



We realize the solution on the amplitudes of the vibrating mass in a forced vibration systems is:

$$y(t) = c_1\,Cos\omega_0 t + c_2\,Sin\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \tag{4.35}$$

Question: What will happen in the case of: $\omega = \omega_0$?

We will observe that the amplitude y(t) in Equation (4.35) turn into situation:

$$y(t) \to \infty$$

Meaning the amplitude of vibration becomes infinity instantly at all times

which is not physically possible

An **alternative solution** needs to be derived for the case of

$$\omega = \omega_0$$

The **resonant vibration** in the situation of:

The frequency of the excitation (applied) force (**ω**)

= The circular frequency (NATURAL FREQUENCY) of the Mass- spring system (**ω_0**)

The Resonant Vibration Analysis

Because we have the situation with $\omega = \omega_0$, the DE in Equation (4.33) now can be Written as:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = F_0\,Cos\,\omega_0\,t \qquad \text{(a)}$$

We observe the complementary solution of Equation (a) remains to be:

$$y_h(t) = c_1\,Cos\,\omega_o t + c_2\,Sin\,\omega_o t$$

which has the same "Cosω_0t" as in the non-homogeneous part of the DE.

Consequently, the particular solution of Equation (a) falls into a "special case" category.

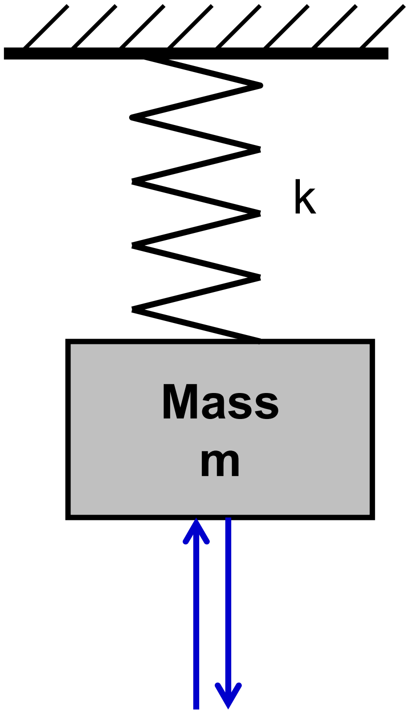
Let us now assume the particular solution to be:

$$y_p(t) = \mathbf{t}\,(A\ Cos\ \omega_0 t + B\ Sin\ \omega_0 t)$$

By following the same procedure as we used in solving non-homogeneous DEs, we get:

A = 0 and $B = \dfrac{F_0}{2m\omega_0}$

Hence $$y_p(t) = \frac{F_0}{2m\omega_0}\,t\,Sin\,\omega_0\,t$$



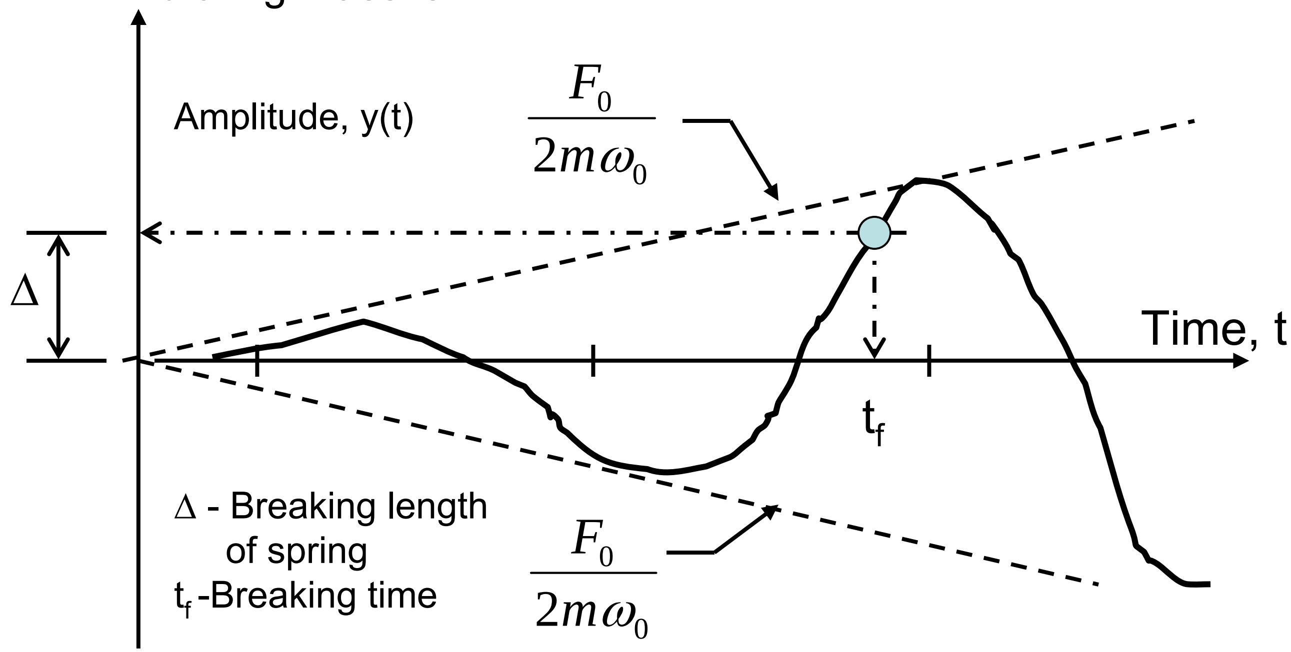
Applied force

$F(t) = F_o Cos\omega_0 t$

The amplitude of the vibrating mass in resonant vibration is:

$$y(t) = c_1 \, Cos\,\omega_0\, t + c_2 \, Sin\,\omega_0\, t + \frac{F_0}{2m\omega_0} \; t \; Sin\,\omega_0\, t \qquad (4.36)$$

Graphical representation of the amplitude fluctuation of the vibrating mass is:



Resonant vibration phenomenon from above graphical illustration:

- **The amplitude of vibration of the mass will increase RAPIDLY with time**
- **The attached spring will soon be "stretched" to break with elongation Δ in a short time at t_f**

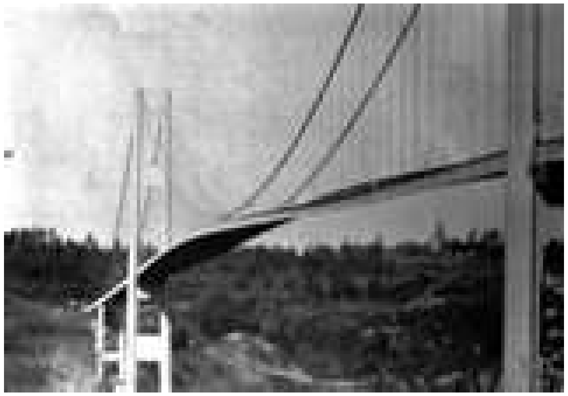
Catastrophic Failure of Tacoma Narrow Bridge
- A classical case of structure failure by Resonant Vibration

- The bridge was located in Tacoma, Washington
- Started building on Nov 23, 1938
- Opened to traffic on July 1, 1940

- The bridge was 2800 feet long, 39 feet wide
- A 42 mph wind blew over the bridge in early morning on November 7, 1940

- The wind provided an external periodic frequency that matched the natural structural frequency of the bridge

- The bridge began to gallop with increasing magnitudes



- Eventual structure failure at about 11 AM

No human life was lost. A small dog was perished because he was too scared to run for his life

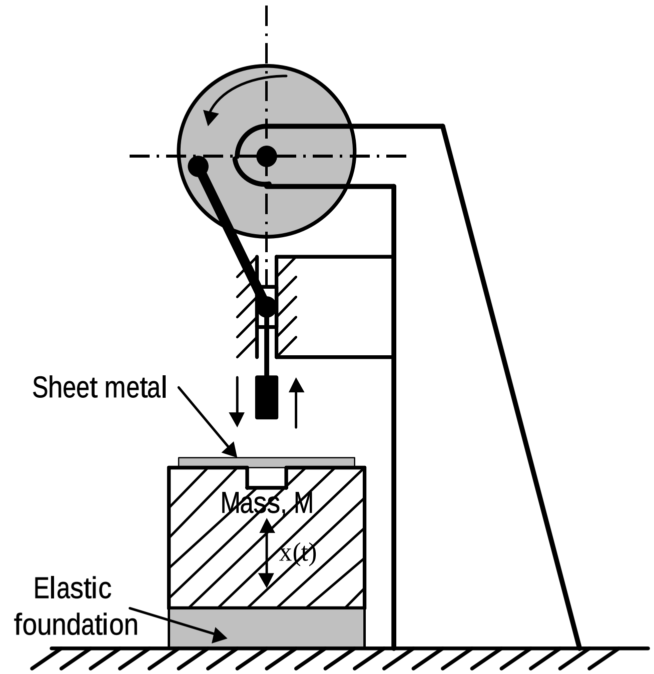
Example 4.9 Resonant vibration of a machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger

The plunger moves vertically up-n-down by a flywheel spinning at constant set speed

The constant rotational speed of the flywheel makes the impact force on the sheet metal, and therefore the supporting base, intermittent and cyclic

The heavy base on which the metal sheet is situated has a mass M = 2000 kg



The force acting on the base follows a function: $F(t) = 2000 \, \mathrm{Sin}(10t)$, in which t = time in seconds

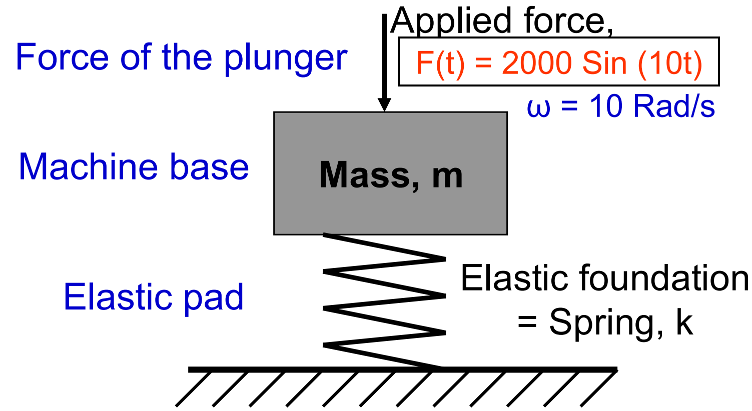
The base is supported by an elastic pad with an equivalent spring constant $k = 2\text{x}10^5$ N/m

Determine the following if the base is initially depressed down by an amount 0.1 m:

(a) The DE for the instantaneous position of the base, i.e., x(t)
(b) Examine if this is a resonant vibration situation with the applied load
(c) Solve for x(t)
(d) Should this be a resonant vibration, how long will take for the support to break at an elongation of 0.3 m?

Solution:

The situation can be physically modeled to be a mass-spring system:



(a) The governing DE from Equation (4.31):

$$2000\frac{d^2x(t)}{dt} + 2x10^5\ x(t) = 2000\,Sin10t \qquad (4.37)$$

with initial conditions:

$$x(0) = 0.1 \text{ m, and } \left.\frac{dx(t)}{dt}\right|_{t=0} = 0 \qquad (4.37a)$$

(b) To check if this is a resonant vibration situation:

Let us calculate the Natural (circular) frequency of the mass-spring system by using Equation (4.16a), or:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2x10^5}{2x10^3}} = 10\ Rad/s$$ = ω, the frequency of the excitation force

So, it is a resonant vibration because $\omega_0 = \omega$

(c) Solution of DE in Equation (4.37):

It is a non-homogeneous DE, so the solution consists two parts:

$$x(t) = x_h(t) + x_p(t) \qquad \text{(a)}$$

By now, we know how to solve for the complementary solution $x_h(t)$ in the form:

$$x_h(t) = c_1\, Cos10t + c_2\, Sin10t \qquad \text{(b)}$$

Because it is a resonant vibration – a special case for solving non-homogeneous 2nd order DEs, the particular solution $x_p(t)$ will take the form:

$$x_p(t) = t\,(A Cos10t + B Sin10t) \qquad \text{(c)}$$

By following the normal procedure of substituting the $x_p(t)$ in Equation (c) into the DE in Equation (4.37), and comparing terms on both sides, we will have the constants A and B in Equation (c) computed as: A = -1/20 and B = 0.

We will thus have the particular solution

$$x_p(t) = -t/20 \qquad \text{(d)}$$

By substituting Equation (b) and (d) into (a), we will have the general solution of Equation (4.37) to be:

$$x(t) = x_h(t) + x_p(t) = c_1\, Cos10t + c_2\, Sin10t - \frac{t}{20} Cos10t \qquad \text{(e)}$$

Apply the two specified initial conditions in Equation (4.37a) into the above general solution will result in the values of the two arbitrary constants:

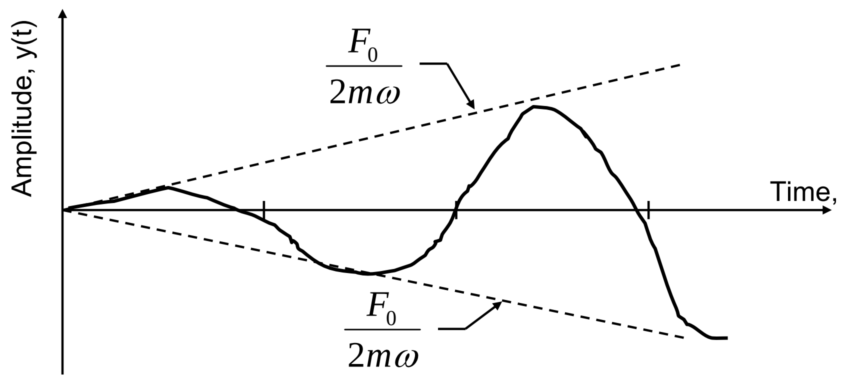
$c_1 = 0.1$ and $c_2 = 1/200$

The complete solution of Equation (4.37) is thus:

$$x(t) = \frac{1}{10} Cos10t + \frac{1}{200} Sin10t - \frac{t}{20} Cos10t \qquad \text{(f)}$$

Graphic representation of x(t) in Equation (f) is similar to the graph on the right with amplitudes increase rapidly with time t.

Physically, the amplitudes are the elongation of the attached spring support



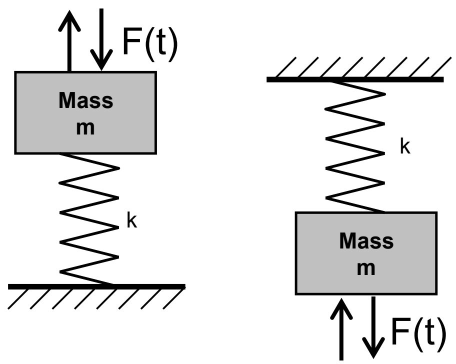
(d) Determine the time to break the elastic support pad:

Since the elastic pad will break at an elongation of 0.3 m, we may determine the time to reach this elongation (t_f) by the following mathematical expression:

$$0.3 = \frac{1}{10} Cos10t_f + \frac{1}{200} Sin10t_f - \frac{t_f}{20} Cos10t_f = \left(\frac{1}{10} - \frac{t_f}{20}\right) Cos10t_f + \frac{1}{200} Sin10t_f$$

Solving for t_f from the above equation leads to T_f = 8 s from the beginning of the resonant vibration

Near Resonant Vibration Analysis



We have learned resonant vibration happens when
The frequency of the applied intermittent forces to the mass (ω)
= The natural frequency of the mass-spring system (ω_0)

There are times when $\omega \neq \omega_0$, but $\omega \approx \omega_0$
Such is the case called "Near Resonant" vibration

Because we have the case $\omega \neq \omega 0$, we could use the solution obtained for the case for $F(t) = F_0$ Cos ωt:

$$y(t) = c_1 Cos\omega_0 t + c_2 Sin\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \qquad (4.35)$$

If we impose the initial conditions:

y(0) = 0 for initial displacement, and

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = 0 \quad \text{for initial velocity}$$

We may determine the arbitrary constants: $c_1 = -\frac{F_0}{m\left(\omega_o^2 - \omega^2\right)}$ and $c_2 = 0$

The complete solution for the DE in Equation (4.33) becomes:

$$y(t) = \frac{F_o}{M\left(\omega_o^2 - \omega^2\right)}\left[Cos(\omega t) - Cos(\omega_o t)\right] \qquad (4.38)$$

By using the expressions for "half-angles" in trigonometry:

$$Cos\alpha + Cos\beta = 2Cos\frac{1}{2}(\alpha+\beta)Cos\frac{1}{2}(\alpha-\beta) \quad \text{and} \quad Cos\alpha - Cos\beta = -2Sin\frac{1}{2}(\alpha+\beta)Sin\frac{1}{2}(\alpha-\beta)$$

Substituting the above relations into Equation (4.38) will lead the following:

$$y(t) = \frac{2F_o}{M\left(\omega_o^2 - \omega^2\right)} Sin\left[(\omega_o + \omega)\frac{t}{2}\right] Sin\left[(\omega_o - \omega)\frac{t}{2}\right] \qquad (4.39)$$

But we have $\omega \approx \omega_0$, hence $\omega_0 - \omega \rightarrow 0$ in Equation (4.39), we thus have the following special relationships:

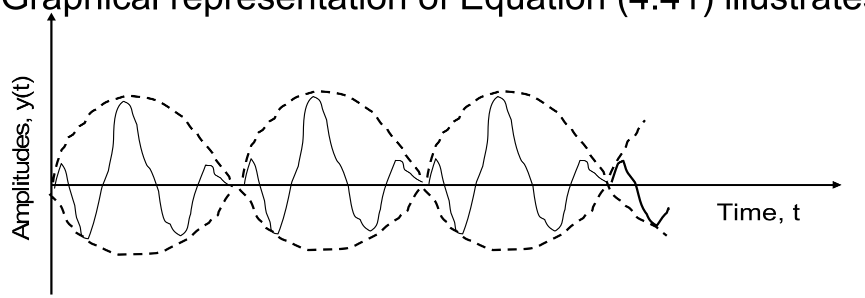
$$\frac{\omega_o + \omega}{2} \approx \omega \quad and \quad \frac{\omega_o - \omega}{2} = \varepsilon$$

in which the circular frequency $\varepsilon << \omega$ (the frequency of the exciting force)

Consequently, the solution in Equation (4.39) can be expressed as:

$$y(t) = \left[\frac{2F_o}{M(\omega_o^2 - \omega^2)}\right] Sin(\varepsilon t)\, Sin(\omega t) \qquad (4.41)$$

Graphical representation of Equation (4.41) illustrates vibration in oscillations with "beats" with:



$$f_b = \frac{\varepsilon}{2\pi}$$ to be the frequency of the beats

$$y(t) = \left[\frac{2F_o}{M(\omega_o^2 - \omega^2)}\right]$$ to be the maximum amplitudes

Near resonant vibration is not usually catastrophic to the structure as resonant vibration but it can cause unwanted disturbance and fatigue failures of structures

Part 4

The Modal Analysis

We realize from vibration analysis of simple mass-spring system that resonant vibration can occur when the frequency of applied force (ω) equals the natural frequency of the mass-spring structure (ω_0)

Resonant vibration can lead to catastrophic failure of the structure, and it should always be avoided by engineers

To avoid such happening, we need to know the natural frequency of the structure, so that we can avoid resonant vibration from happening to the structure by not applying any cyclic forces to the structure at frequencies that coincide the natural frequencies of the structure

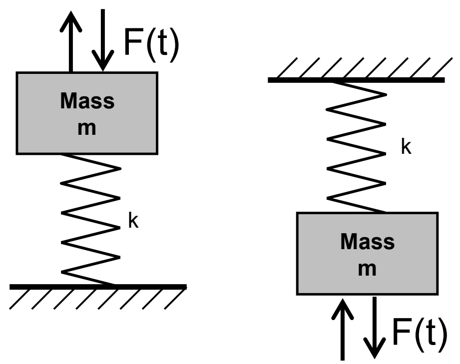
MODAL ANALYSIS is a process of determining the natural frequency or frequencies of a machine or structure

For simple mass-spring systems with the mass being attached or supported by a single spring, the mass vibrates in one-degree-of freedom (because the motion of the mass is prompted by a single spring force)

One degree-of-freedom system has only ONE MODE of natural frequency – one natural frequency, ω_0

For structures of complex geometry subjected to complex loading, there exists Infinite (∞) degree-of-freedom, and thus infinite number of natural frequencies – calling Mode 1, 2, 3, ……….natural frequencies, expressed by: **ω_n: ω_1, ω_2, ω_3, .ω_∞**

Every effort should be made not to apply any intermittent cyclic forces with frequency coinciding ANY of the natural frequency in any mode of the structure



The natural frequency of the simple mass-spring systems is:

$$f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (4.17)$$

For structures of complex geometry and loading conditions, the elastic support can no longer be represented by a single spring with spring constant k, and the mass is distributed in the structure according to its geometry. In such cases the natural frequencies are determined by the following generalized formula:

$$\omega_n = \sqrt{\frac{[k]}{[m]}}$$

Mode number n = 1, 2, 3, ….,n

where [k] and [m] are respective "stiffness matrix" and "mass matrix" of the structure

These matrices are obtained by numerical analyses, such as finite element stress analysis

MODAL ANALYSIS is an essential analysis for any machine or structure expected to be subject to time-varying loads