Signal Classification

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Signal

- A signal is a pattern of variation that carry information.
- Signals are represented mathematically as a function of one or more independent variable
  - A picture is brightness as a function of two spatial variables, $x$ and $y$.
- In this course signals involving a single independent variable, generally refer to as a time, $t$ are considered. Although it may not represent time in some specific applications.
- A signal is a real-valued or scalar-valued function of an independent variable $t$. 
Example of signals

- Electrical signals like voltages, current and EM field intensity in circuit
- Acoustic signals like audio or speech signals (analog or digital)
- Video signals like intensity variation in an image
- Biological signal like sequence of bases in gene
- Noise which will be treated as unwanted signal
- …
Signal classification

- Continuous-time and Discrete-time
- Energy and Power
- Real and Complex
- Periodic and Non-periodic
- Analog and Digital
- Even and Odd
- Deterministic and Random
A continuous-time signal

Continuous-time signal $x(t)$, the independent variable, $t$ is Continuous-time. The signal itself needs not to be continuous.

```matlab
x=0:0.1:5;
y=2*sin(x.^2).*exp(-x);
plot(x,y)
```
A piecewise continuous-time signal

\[ x(t) = \begin{cases} 
  0; & 0 \leq t \leq 10 \\
  1; & 10 < t \leq 20 \\
 -1; & 20 \leq t \leq 25 \\
  2; & 25 < t \leq 30 
\end{cases} \]
Sampling

- A discrete signal can be derived from a continuous-time signal by sampling it at a **uniform rate**.
- If $\tau$ denotes the sampling period and $n$ denotes an integer that may assume positive and negative values, sampling a continuous-time signal $x(t)$ at time $t = n\tau$ yields a sample of value $x(n\tau)$.
- For convenience, a discrete-time signal is represented by a sequence of numbers: $x(n\tau)$.
- We write $x[n] = x(n\tau)$ \forall n$.
- Such a sequence of numbers is referred to as a **time series**.
A discrete-time signal

A discrete signal $x[n]$ is defined only at discrete instances. Thus, the independent variable has discrete values only.

t = 0:0.1:5;
x = 2*sin(t.^2).*exp(-t);
stem(t,x)
A piecewise discrete-time signal

\[ x[n] = \begin{cases} 
0; & 0 \leq n \leq 10 \\
1; & 10 < n \leq 20 \\
-1; & 20 \leq n \leq 25 \\
2; & 25 < n \leq 30 
\end{cases} \]
Real and Complex

- A value of a complex signal $x(t)$ is a complex number
  
  $x(t) = x_1(t) + jx_2(t)$

  Where $j = \sqrt{-1}$  \[ \text{Re}\{x(t)\} = x_1(t) \quad \text{Im}\{x(t)\} = x_2(t) \]

- The complex conjugate, $x^*(t)$ of the signal $x(t)$ is;
  
  $x^*(t) = x_1(t) - jx_2(t)$

  \[ \text{Re}\{x(t)\} = \frac{1}{2} [x(t) + x^*(t)] \quad \text{Im}\{x(t)\} = \frac{1}{2j} [x(t) - x^*(t)] \]

- Magnitude or absolute value
  
  $|x(t)| = \sqrt{x_1^2(t) + x_2^2(t)} = \sqrt{x(t)x^*(t)}$

- Phase or angle
  
  $\tan \theta = \frac{x_2(t)}{x_1(t)}$
Complex-Valued Signal Symmetry

- For a complex-valued signal \( x(t) = x_1(t) + j x_2(t) \)

- is said to be conjugate symmetric if it satisfies the condition

\[
 x(-t) = x^*(t) \\
 x^*(t) = x_1(t) + j x_2(t)
\]

- \( x_1(t) \) is the real part and \( x_2(t) \) is the imaginary part;

- \( j \) is the square root of -1

What is square root of \( j \)??
Periodic and Non-periodic

A signal $x(t)$ or $x[n]$ is a periodic signal if

$$x(t) = x(t + mT_o); \quad \forall t, \quad m = \text{any integer}$$

$$x[n] = x[n + mN_o]; \quad \forall n, \quad m = \text{any integer}$$

Here, $T_o$ and $N_o$ are fundamental period, which is the smallest positive values when $m = 1$

Example:

$$x(t) = A \cos(\omega_o t + \theta); \quad -\infty < t < \infty; \quad f = \frac{\omega_o}{2\pi}$$

$$x(t + T_o) = A \cos(\omega_o [t + mT_o] + \theta) = A \cos(\omega_o t + \theta)$$

$$\Rightarrow T_o = \frac{2\pi}{\omega_o}; \quad m = 0, 1, 2, \ldots$$
Analog and Digital

- **Digital signal** is a discrete-time signal whose values belong to a defined set of real numbers \( \{a_1, a_2, \ldots, a_N\} \)

\[
x[n] = x(t_n) = a_i; \quad 1 \leq i \leq N
\]

- **Binary signal** is a digital signal whose values are 1 or 0

\[
x[n] = x(t_n) = 0 \text{ or } 1; \quad \forall n
\]

- **Analog signal** is a non-digital signal
Even and Odd

- **Even Signals**
  The continuous-time signal $x(t)$ or discrete-time signal $x[n]$ is an even signal if it satisfies the condition
  $$x(t) = x(-t); \quad \forall t \quad x[n] = x[-n]; \quad \forall n$$
  - Even signals are **symmetric** about the vertical axis

- **Odd Signals**
  The signal is said to be an odd signal if it satisfies the condition
  $$x(t) = -x(-t); \quad \forall t \quad x[n] = -x[-n]; \quad \forall n$$
  - Odd signals are **anti-symmetric (asymmetric)** about the time origin
Even and Odd signals: Facts

- Product of 2 even or 2 odd signals is an **even signal**
- Product of an even and an odd signal is an **odd signal**
- Any signal (continuous and discrete) can be expressed as **sum of an even and an odd signal**:

\[
x(t) = x_e(t) + x_o(t); \quad x[n] = x_e[n] + x_o[n]
\]

\[
x_e(t) = \frac{x(t) + x(-t)}{2}; \quad x_o(t) = \frac{x(t) - x(-t)}{2}
\]

\[
x_e[n] = \frac{x[n] + x[-n]}{2}; \quad x_o[n] = \frac{x[n] - x[-n]}{2}
\]
Deterministic and Random signal

- A signal is **deterministic** whose future values can be predicted accurately.
- Example:
  - A signal is **random** whose future values can NOT be predicted with complete accuracy.

- Random signals whose future values can be statistically determined based on the past values are **correlated signals**.
- Random signals whose future values can NOT be statistically determined from past values are **uncorrelated signals** and are more random than correlated signals.
Two ways to describe the randomness of the signal are:

- **Entropy:**
  This is the natural meaning and mostly used in system performance measurement. It’s the unpredictability in a random variable, usually measured in bits (or its log equiv.)

- **Correlation:**
  This is useful in signal processing by directly using correlation functions. A correlation function is a statistical correlation between random variables at two different points in space or time, usually as a function of the spatial or temporal distance between the points.
Deterministic and Random signal (example)

- The signal $x_1(t)$ and $x_2(t)$ shown below constitute the real and imaginary parts of a complex-valued signal $x(t)$.
- What form of symmetry does $x(t)$ have?

Answer: Conjugate Symmetric $x(-t) = x^*(t)$.
Power and Energy in a Physical System

- The instantaneous power
  \[ P(t) = v(t)i(t) = \frac{1}{R}v^2(t) \]

- The total energy
  \[ \int_{t_1}^{t_2} P(t)\,dt = \int_{t_1}^{t_2} v(t)i(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt \]

- The average power
  \[ \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} P(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt \]
Energy signals

- By definition, the total energy over the time interval $t_1 < t < t_2$ in a continuous-time signal $x(t)$ is:
  $$E = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

- Denote the magnitude of the (possibly complex) ‘amplitude’ $|x(t)|$

- The total energy contained in a continuous-time signal $x(t)$ is
  $$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

- An energy signal is a signal with finite energy and zero average power.
  $$0 < E = \int_{t_1}^{t_2} |x(t)|^2 \, dt < \infty \quad P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = 0$$

- By definition, the total energy over the time interval $n_2 \leq n \leq n_1$ in a discrete-time signal $x[n]$ is:
  $$E = \sum_{n=n_1}^{n_2} |x[n]|^2$$
Power signals

- The time average power over the time interval \( t_1 < t < t_2 \) in a continuous-time signal is:
  \[
P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt
\]

- A power signal is a signal with infinite energy but finite average power:
  \[0 < P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)| dt < \infty \quad E = \int_{t_1}^{t_2} |x(t)|^2 dt \to \infty\]

- The time average power over the time interval \( n_2 \leq n \leq n_1 \) in a discrete-time signal is:
  \[
P = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2
\]

- The square root of the average power \( \sqrt{P} \) of a power signal is what usually defined as the RMS value of the signal.
Energy and Power Signals

- \( x(t) \) is a continuous power signal if:
  \[
  0 < \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty
  \]
  (finite average power, e.g. sine wave)

- \( x[n] \) is a discrete power signal if:
  \[
  0 < \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty
  \]

- \( x(t) \) is a continuous energy signal if:
  \[
  0 < \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty
  \]
  (finite energy, e.g. \( \sin(t) \cdot \exp(-t) \))

- \( x[n] \) is a discrete energy signal if:
  \[
  0 < \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty
  \]
Example 1:
The signal $x(t)$ is given below is energy signal, power signal or neither? Explain.

$$x(t) = 3\sin(2\pi t) \quad -\infty < t < \infty$$

This a periodic signal, so it must be a power signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 dt = 9\int_{-\infty}^{\infty} \frac{1}{2} |1 - \cos(4\pi t)| dt$$

$$E = 9\int_{-\infty}^{\infty} \frac{1}{2} dt - 9\int_{-\infty}^{\infty} \cos(4\pi t) dt = \frac{t}{2} \bigg|_{-\infty}^{\infty} + \frac{9\sin 4\pi t}{4\pi} \bigg|_{-\infty}^{\infty} = \infty \quad J$$

The period of signal $x(t)$ is $T = 2\pi / 2\pi = 1$ second, then the power

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2} \int_{-1}^{1} |3\sin(2\pi t)|^2 dt = \frac{9}{4}\int_{-1}^{1} (1 - \cos 4\pi t) dt = \frac{9}{2}$$

This periodic signal as expected is power signal, because its energy is infinite and is power is finite.
Example 1: continues

The average power of this signal is as expected

\[ P_{\text{ave}} = \frac{A\text{mlitude square}}{2} = \frac{(3)^2}{2} \]
Example 2:
The signal $x(t)$ is given below is energy signal, power signal or neither? Explain.

This signal is an energy signal, because its power is zero and its energy is finite.

\[
P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{1} 3^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \cdot 9 \cdot 1 = \lim_{T \to \infty} \frac{9}{2T} = 0
\]

\[
E = \int_{-T}^{T} |x(t)|^2 \, dt = \int_{0}^{1} 3^2 \, dt = 9 \cdot 1 = 9
\]
Example 3: The signal \( x[n] \) is given below is energy or power signal. Explain.

This signal is an energy signal:

\[
P = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{0}^{2} 3^2 = \lim_{N \to \infty} \frac{27}{2N + 1} = 0
\]

\[
E = \sum_{n=n_1}^{n_2} |x[n]|^2 = \sum_{0}^{2} 3^2 = 27
\]
Power and Energy

- Example 4:
  - The signal $x(t)$ is given below is energy or power signal? Explain.
    $$x(t) = 5e^{-2|t|}, \quad -\infty < t < \infty$$
  - The total energy of the signal.
    $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} 25e^{-2|t|} dt = 25 \int_{-\infty}^{0} e^{4t} dt + \int_{0}^{\infty} e^{-4t} dt$$
    $$E = \frac{25}{4} [e^{4t}]_{-\infty}^{0} + \frac{25}{4} [e^{-4t}]_{0}^{\infty} = \frac{25}{4} + \frac{25}{4} = 50$$
  - The average power of the signal is
    $$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} 5e^{-2|t|} dt$$
    $$P = \frac{25}{4} \lim_{T \to \infty} \frac{1}{T} [e^{4t}]_{-\frac{T}{2}}^{0} + \frac{25}{4} \lim_{T \to \infty} \frac{1}{T} [e^{-4t}]_{0}^{\frac{T}{2}} = \frac{25}{4} \lim_{T \to \infty} \frac{1}{T} [1 - e^{-2T}] + \frac{25}{4} \lim_{T \to \infty} \frac{1}{T} [e^{-2T} - 1] = 0$$
  - The signal $x(t)$ is an energy signal because its energy is finite and its average power is zero.
Power and Energy

Example 5:
The signal \( x(t) \) is given below is energy or power signal? Explain.

\[
x(t) = \begin{cases} 
\frac{1}{\sqrt{t}}, & t > 1 \\
0, & t \leq 1 
\end{cases}
\]

The total energy of the signal.

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{1}^{\infty} \frac{1}{t} \, dt = \ln [t]_{1}^{\infty} = \infty - 0 = \infty
\]

The average power of the signal is

\[
P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \frac{1}{t} \, dt = \lim_{T \to \infty} \left( \frac{1}{T} \ln \left[ \frac{T}{2} \right] \right)
\]

\[
P = \lim_{T \to \infty} \left( \frac{1}{T} \ln \left[ \frac{T}{2} \right] - \frac{1}{T} \ln [1] \right) = \lim_{T \to \infty} \left( \frac{1}{T} \ln \left[ \frac{T}{2} \right] \right) = \lim_{T \to \infty} \left( \frac{\ln \left[ \frac{T}{2} \right]}{T} \right)
\]
Example 5 (cont’d)
Using L’hopital’s rule, the power of the signal is zero. That is

\[
P = \lim_{T \to \infty} \left( \ln \left( \frac{T}{2} \right) \right) = \lim_{T \to \infty} \left( \frac{2}{T} \right) = 0
\]

This signal is not an energy signal. However, it is also not a power signal since its average power is zero.
Taylor series and Euler relation (FYI)

- General Taylor series
  \[ f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \]

- Expanding sine and cos
  \[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \]
  \[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \]

- Expanding exponential
  \[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \]
  \[ e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} + \ldots \]

- Euler relation
  \[ e^{jx} = \left(1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\right) + j \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = \cos x + j \sin x \]