## CHAPTER 4

# Beam Element

## 4.1 Introduction

Beam element is a very versatile line-element, it has six degrees of freedom at each node, which include, translations and rotations along the x, y, and z directions, respectively. Figure 4.1 shows the positive directions of these displacements.

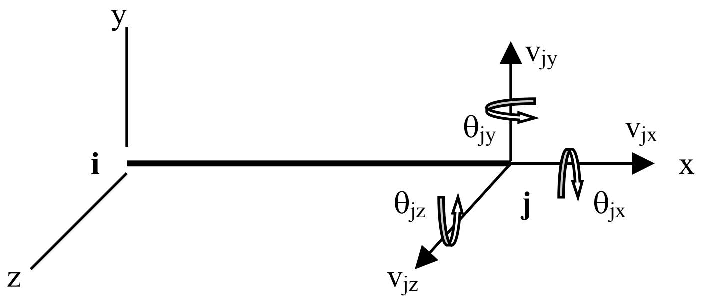


*Figure 4.1 Beam Element with six degrees of freedom at each node*

Beam element is employed to simulate a slender structure that has an uniform cross section. The element is unsuitable for structures that have complex geometry, holes, and points of stress concentration.

The stiffness constant of a beam element is derived by combining the stiffness constants of a beam under pure bending, a truss element, and a torsion bar. Thus, a beam element can represent a beam in bending, a truss element, and a torsion bar. In FEA it's a common practice to use beam elements to represent all or any of these three loads.

We will derive the element stiffness equation for a beam element by first deriving the stiffness equation of a beam in bending, and then superimposing the stiffness of a truss and a torsion bar element.

# 4.1 Derivation of a Stiffness Equation for a Beam Element Under Pure Bending

A beam, such as, a cantilever beam, under pure bending (without axial loads or torsional loads), has two-degrees of freedom at any point, transverse deflection v and rotation $\theta$, as shown in Figure 4.2.

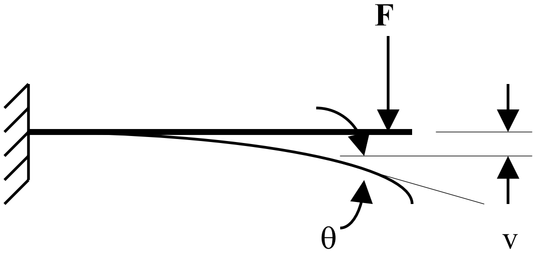


*Figure 4.2 Cantilever Beam with it's DOF, v and* $\theta$

A beam element has a total of four degrees of freedom, two at each node. Since there are four degrees of freedom, the size of the stiffness matrix of a beam element has the size 4 x 4.

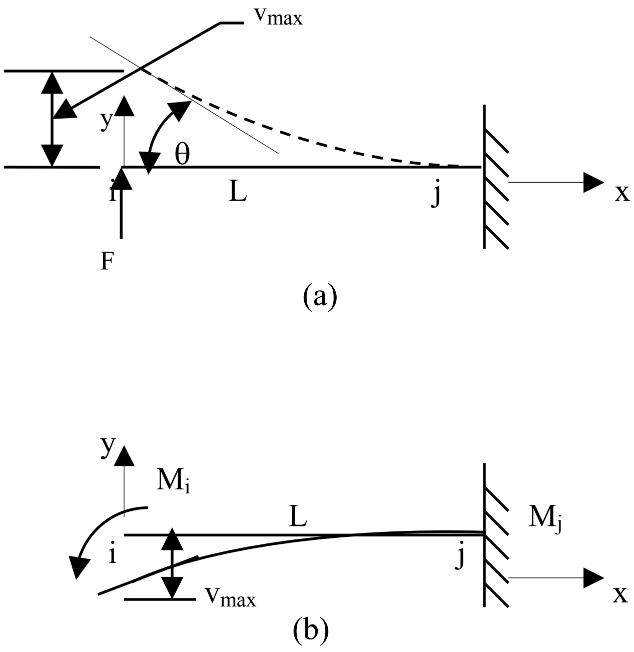
We will derive the stiffness matrix equation using a simple method, known as Stiffness Influence Coefficient Method. In this procedure, a relationship between force and the coefficients that influence stiffness is established. For a beam element, these coefficient consist of: the modulus of elasticity, moment of inertia, and length of the element. For a two-node beam element, there are two deflections and two rotations, namely, $v_1$, $\theta_1$, $v_2$, and $\theta_2$. Force and influence coefficient relationship is established by setting each of the four deflection values to unity, with the remaining deflection values equal to zero. The procedure follows.

Consider a beam element, loaded in such a way that it has the deflection values: $v_i = 1$, $\theta_i = 0$, $v_j = 0$, $\theta_j = 0$

*Figure 4.3 Beam Element*

The above deflections can be produced by a combination of load conditions, shown in figure 4.4.

The following deflection relationships for loading of Figures 4.4 (a) and (b) can be found in any Machine Design Handbook, and is given as,



(a)

$v_{max} = (FL^3)/(3EI)$

$\theta = -(FL^2)/(2EI)$

(b)

$v_{max} = -(ML^2)/(2EI)$

$\theta = (ML)/(EI)$

*Figure 4.4*

Applying these relationships to the beam of Figure 4.3, we get,

$1 = v_i = (v_i)_F + (v_i)_M$

$$1 = v_i = (F_i L^3)/3EI - (M_i L^2)/2EI \qquad (4.1)$$

and $\theta = 0 = (\theta)_F + (\theta)_M$

$$0 = -(F_i L^2)/2EI + (M_i L)/EI \qquad (4.2)$$

Solving Equations (4.1) and (4.2), we get,

$F_i = (12EI)/L^3$ (A)

$F_j = -F_i = -(12EI)/L^3$ (B)

$M_i = (6EI)/L^2$ (C)

From Figure 4. 4 (a) and (b),

$$\begin{aligned} M_j &= F_i L - M_i \\ &= (12EI)/L^2 = (6EI)/ L^2 \\ &= (6EI)/ L^2 \qquad\qquad (D) \end{aligned}$$

Writing equations (A) through (D) in a matrix form we get,

$$\begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} = \begin{pmatrix} (12EI)/L^3 \\ (6EI)/ L^2 \\ -(12EI)/ L^3 \\ (6EI)/ L^2 \end{pmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{pmatrix} (12EI)/ L^3 & 0 & 0 & 0 \\ (6EI)/ L^2 & 0 & 0 & 0 \\ -(12EI)/ L^3 & 0 & 0 & 0 \\ (6EI)/ L^2 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using a similar procedure and setting the following deflection values:

$v_i = 0$, $\theta_i = 1$, $v_j = 0$, $\theta_j = 0$, we get,

$$\begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} = \begin{pmatrix} (6EI)/L^2 \\ (4EI)/ L \\ -(6EI)/ L^2 \\ (2EI)/ L \end{pmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{pmatrix} 0 & (6EI)/ L^2 & 0 & 0 \\ 0 & (4EI)/ L & 0 & 0 \\ 0 & -(6EI)/ L^2 & 0 & 0 \\ 0 & (2EI)/ L & 0 & 0 \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \qquad (4.6)$$

Similarly, setting $v_j = 1$ and , $\theta_j = 1$, respectively, and keeping all other deflection values to zero, we get the final matrix as,

$$\begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} = \begin{pmatrix} (12EI)/L^3 & (6EI)/ L^2 & -(12EI)/ L^3 & (6EI)/ L^2 \\ (6EI)/ L2 & (4EI)/ L & -(6EI)/ L^2 & (2EI)/ L \\ -(12EI)/ L^3 & -(6EI)/ L^2 & (12EI)/L^3 & -(6EI)/ L^2 \\ (6EI)/ L^2 & (2EI)/ L & -(6EI)/ L^2 & (4EI)/ L \end{pmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \qquad (4.7)$$

Note that, the first term on the RHS of the above equation is the stiffness matrix and the second term is the deflection. In the case where deflections are other than unity, the above equation will provide an element equation for a beam (in bending), which can be written as,

$$\begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix} = \begin{pmatrix} (12EI)/L^3 & (6EI)/L^2 & -(12EI)/L^3 & (6EI)/L^2 \\ (6EI)/L2 & (4EI)/L & -(6EI)/L^2 & (2EI)/L \\ -(12EI)/L^3 & -(6EI)/L^2 & (12EI)/L^3 & -(6EI)/L^2 \\ (6EI)/L^2 & (2EI)/L & -(6EI)/L^2 & (4EI)/L \end{pmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \qquad (4.7)$$

Where $F_i$, $M_i$, $F_j$, $M_j$ are the loads corresponding to the deflections $v_i$, $\theta_i$, $v_j$, $\theta_j$.

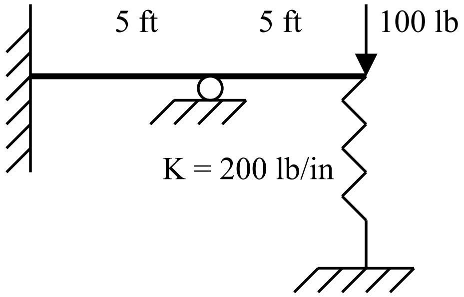
Equation (4.7) is the equation of a beam element, which is under pure bending load (no axial or torsion loads). The stiffness matrix is a 4 x 4, symmetric matrix. Using this equation, we can solve problems in which several beam elements are connected in an uni-axial direction. The assembly procedure is identical to the truss elements. However, if the beam elements are oriented in more than one direction, we will have to first transform the above equation (4.7) in to a global stiffness matrix equation (analogues to the procedure used for truss elements).

For a beam element, transformation of a local stiffness matrix into a global equation involves very complex trigonometric relations, and therefore, we will defer the derivations at this time. However, Equation (4.7) can be used for solving a beam problem, loaded under bending loads. In order to understand the application of this equation, we will apply it to solve some statically indeterminate problems.

### Example 1

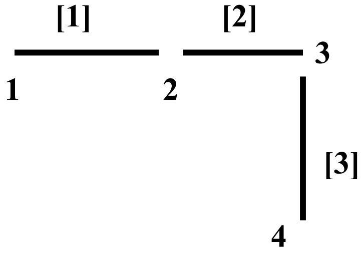
For the beam shown, determine the displacements and slopes at the nodes, forces in each element, and reactions at the supports.

$E = 1.4 \times 10^6$ psi, $I = 2.4$ in$^4$



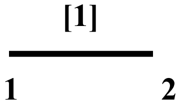
### Solution

The beam structure is descritized into three elements and 4-nodes, as shown.



First, we will find the element stiffness matrix for each element, next we will assemble the stiffness matrices, apply the boundary conditions, and finally, solve for node deflection. Internal forces and reactions are calculated by back-substituting the deflections in the structural equation.

<u>Element 1</u>

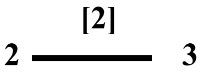


$EI/L^3 = (1.4 \times 10^6) \times (2.4)/(5\times12)^3 = 15.55$

The general equation of a stiffness matrix is given as,

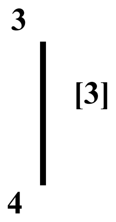
$$[K_e]^{(1)} = (EI/L^3) \left(\begin{array}{cccc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right) \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

Element 2



$$[K_e]^{(1)} = (EI/L^3) \left(\begin{array}{cccc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right) \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Element 3



$$[K_e]^{(3)} = \left(\begin{array}{cc} K & -K \\ -K & K \end{array}\right) \begin{array}{c} v_3 \\ v_4 \end{array}$$

To get the global stiffness matrix, we will use the same procedure used for assembling truss element stiffness equations. In terms of E, L, and I the assembled global stiffness matrix is,

$$\begin{array}{c|ccccccc|}
 & v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \\
v_1 & 12 & 6L & -12 & 6L & 0 & 0 & 0 \\
\theta_1 & & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
v_2 & & & 24 & 0 & -12 & 6L & 0 \\
\theta_2 & & & & 8L^2 & -6L & 2L^2 & 0 \\
v_3 & & & & & 12+K' & -6L & -K' \\
\theta_3 & & & & & & 4L^2 & 0 \\
v_4 & \text{SYMMETRY} & & & & & & K'
\end{array} \times (EI)/(L^3)$$

Where $K' = (K) \times [L^3 / (EI)]$

Our next step is to write the structural equation; however, we can reduce the size of the stiffness matrix by applying the given boundary conditions:

$v_1 = \theta_1 = 0$ node 1 is fixed

$v_2 = 0$ node 2 has no vertical deflection, but it's free to rotate.

$V_4 = 0$ node 4 is fixed.

The reduced stiffness matrix is

$$K_G = EI/(L^3) \begin{pmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+K' & -6L \\ 2L^2 & -6L & 4L^2 \end{pmatrix}$$

Substituting the values of E, L, and I the structural equation can be written as,

$$\begin{Bmatrix} 0 \\ -100 \\ 0 \end{Bmatrix} = (15.55) \begin{pmatrix} 1152 & -72 & 288 \\ -72 & 16.11 & -72 \\ 288 & -72 & 576 \end{pmatrix} \begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Solving, we get

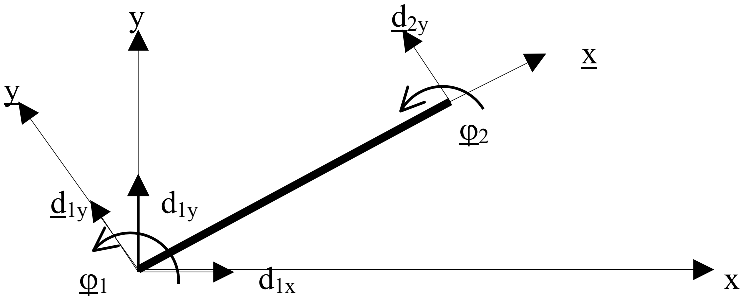
$\theta_2 = -\ 0.0032$ rad

$v_3 = -\ 0.4412$ in

$\theta_3 = -0.0095$ rad

# 4.2 Arbitrarily Oriented 2-D Beam Element

The stiffness equation for an arbitrarily oriented beam element can be derived with a procedure similar to the truss element.



$\underline{d}_{1y} = d_{1y} \cos\theta - d_{1x} \sin\theta = d_{1y}\, c - d_{1x}\, s$

$\underline{d}_{2y} = d_{1y} \cos\theta – d_{2x} \sin\theta = d_{2y}\, c – d_{2x}\, s$

and $\underline{\varphi}_1 = \varphi_1$, $\underline{\varphi}_2 = \varphi_2$

Note: The underscored terms represent local coordinate values. Thus, $\underline{x}$ and $\underline{y}$ are local coordinates and x and y are global coordinates.

The above equations can be written in a matrix form,

$$\begin{Bmatrix} \underline{d}_{1y} \\ \underline{\varphi}_1 \\ \underline{d}_{2y} \\ \underline{\varphi}_2 \end{Bmatrix} = \begin{pmatrix} -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \varphi_1 \\ d_{2x} \\ d_{2y} \\ \varphi_2 \end{Bmatrix}$$

Let $T = \begin{pmatrix} -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, the transformation matrix.

Thus, $\underset{\text{Local}}{\{\underline{d}\}} = [T]\ \underset{\text{Global}}{\{d\}}$

Note that angle φ is independent of the coordinate systems, and $\underline{\varphi}_1 = \varphi_1$, $\underline{\varphi}_2 = \varphi_2$

As derived in the case of the truss element, relationship between local and global stiffness matrices is given as

$$[k_g] = [T]\,[\underline{k}]\,[T]$$

Where, $[k_g]$ = Global stiffness matrix of an element

$[T]$ = Transformation matrix

$[\underline{k}]$ = Local stiffness matrix of the element

Substituting the values of [T] and $[\underline{k}]$, we get the global equation of a beam element oriented arbitrarily at an angle $\theta$ as,

$$k = EI/L \begin{pmatrix} 12S^2 & -12SC & -6LS & -12S^2 & -12SC & -6LS \\ & 12C^2 & 6LC & 12SC & -12C^2 & 6LC \\ & & 4L^2 & 6LS & -6LC & 2L^2 \\ & & & 12S^2 & -12SC & 6LS \\ \text{Symmetry} & & & & 12C^2 & 4L^2 \end{pmatrix}$$

This is the equation of a beam element (without axial or torsional load, and oriented at an angle $\theta$.

Also, S = $\sin\theta$, C = $\cos\theta$ in the above equation.

## 4.3 Beam Element with Combined Bending and Axial loads

First, we will derive the stiffness matrix in local coordinates and then convert it in to global coordinates.

### 4.3.1 Stiffness matrix of a beam element with bending and axial loads in local coordinates

The stiffness equation for the combined bending and axial load can be written by superimposing the axial stiffness terms over the bending stiffness.

For axial loading, the structural equation is,

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = AE/L^3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

And for bending loading, the structural equation is,

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = AE/L^3 \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix} \begin{Bmatrix} d_{1y} \\ \varphi_1 \\ d_{2y} \\ \varphi_2 \end{Bmatrix}$$

Therefore, the combined loading equation is

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{Bmatrix} = \begin{pmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12\,C_2 & 6C_2L & 0 & -12\,C_2 & 6C_2L \\ 0 & 6\,C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12\,C_2 & -6C_2L & 0 & 12\,C_2 & -6C_2L \\ 0 & 6\,C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{pmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \varphi_1 \\ d_{2x} \\ d_{2y} \\ \varphi_2 \end{Bmatrix}$$

$$\text{And,}\,[k] = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 4C_2L^2 \end{bmatrix}$$

Where, $C_1 = AE/L$, and $C_1 = EI/L^3$

## 4.3.2 Transformation matrix for combined Bending and Axial loading.

For the axial loading, the relationship between the local and global coordinates was derived earlier, as

$$\hat{d}_{1x} = d_{1x}\cos\theta + d_{1y}\sin\theta$$
$$= d_{1x}C + d_{1y}S$$
$$\hat{d}_{2x} = d_{2x}C + d_{2y}S$$

Also, for bending load, derived previously,

$$\hat{d}_{1y} = d_{1y}C - d_{1x}S$$
$$\hat{\phi}_1 = \phi$$
$$\hat{d}_{2y} = d_{2y}C - d_{2x}S$$
$$\hat{\phi}_2 = \phi$$

Therefore, the relationship for the combined bending and axial loading can be written as

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

Or,

$$\left\{\hat{d}\right\} = [T]\{d\}$$

$$\text{Where, } [T]=\begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

## 4.3.3 2-D Beam Element Equation for Combined Loading – Axial and Bending – at an Arbitrary Orientation θ

Substituting the values of $\left[\hat{K}\right]$ and [T] into the equation $\left[K\right]=[T]^T[\hat{K}][T]$, we get

$$K=\frac{E}{L}\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & \textit{symmetry} & & & AS^2+\frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ & & & & & 4I \end{bmatrix}$$

## 4.4 2-D Beam Element with combined loading Bending, Axial, and Torsion (θ = 0)

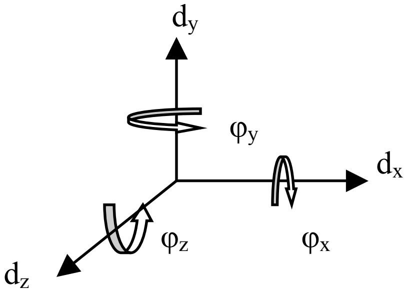
The torsional loads are $m_{1x}$ and $m_{2x}$, and the corresponding deflections are, $\phi_{1x}$ and $\phi_{2x}$

The torsional structural equation is:

$$\begin{pmatrix} m_{1x} \\ m_{1x} \end{pmatrix} = JG/L \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{1x} \\ \varphi_{2x} \end{pmatrix}$$

These terms can be superimposed on the stiffness equation derived previously for the combined bending and axial loads.

3-D Beam Element:



A 3-D beam element has 6 DOF at each node, and 12 DOF for each element. The stiffness matrix can be derived by super-imposing the axial, bending, and torsion loadings in the XY, XZ, and YZ planes. The equation is,

$$
K=\begin{array}{c}
\begin{array}{cccccccccccc}
\hat{d}_{1x} & \hat{d}_{1y} & \hat{d}_{1z} & \hat{\phi}_{1x} & \hat{\phi}_{1y} & \hat{\phi}_{1z} & \hat{d}_{2x} & \hat{d}_{2y} & \hat{d}_{2z} & \hat{\phi}_{2x} & \hat{\phi}_{2y} & \hat{\phi}_{2z}
\end{array}\\
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
\end{array}
$$