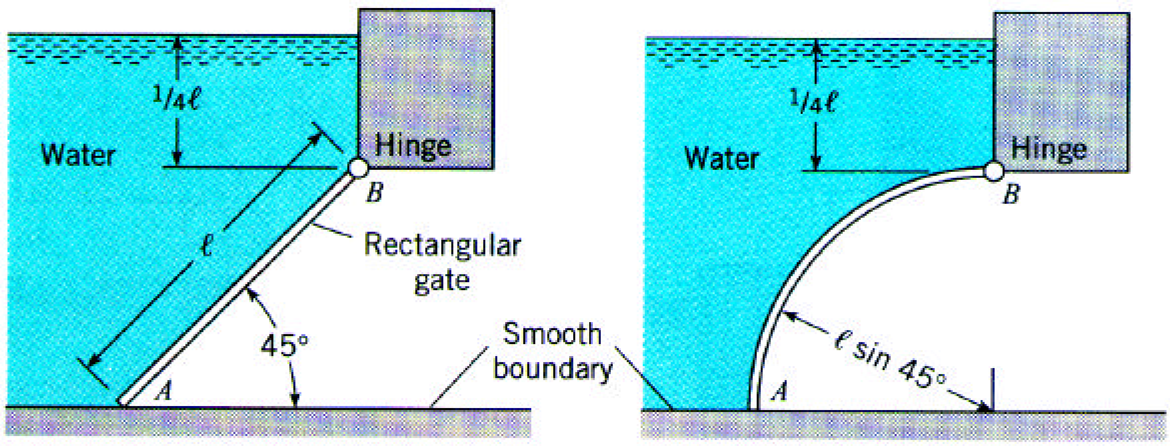
HYDROSTATICS

Example Problem: Hydrostatic Forces on Submerged Surfaces

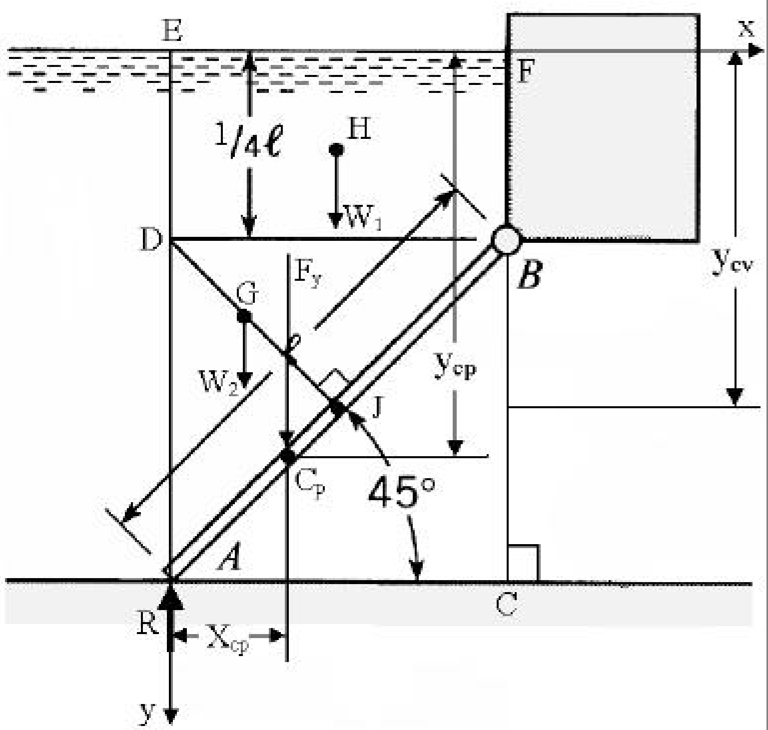
A plane rectangular gate is submerged in water as shown in figure (a). What is the magnitude of the reaction force at A? For the cylindrical gate in figure (b), will the magnitude of the reaction at A be greater than, less than, or the same as that for the plane gate?



(a) Plane gate (b) Curved gate

Solution

Follow the steps for calculating the hydrostatic force on a submerged surface:



1. The vertical projection of the gate AB is the rectangle BC which has the same width (w) as the gate itself.
2. The centroid of BC is located at a depth

$$y_{cv} = \frac{l}{4} + \frac{BC}{2}$$

$$BC = l\frac{\sqrt{2}}{2}$$

combining these two eqs. we get $y_{cv} = \frac{l}{4}\left(1+\sqrt{2}\right)$

3. It follows that the hydrostatic pressure at the centroid of the vertical projection (as well as at the centroid of the gate itself) is

$$p_c = g\ y_{cv} = g\ \frac{l}{4}\left(1+\sqrt{2}\right)$$

4. The area of the vertical projection of the gate is: $A_V = (BC)w = lw\frac{\sqrt{2}}{2}$

5. The horizontal component of the hydrostatic force can now be calculated:

$$F_x = p_c A_V = 0.43g\ l^2\ w$$

6. The volume of the water directly over the surface of the gate can be calculated by splitting it into convenient shapes, such as the prism ABD and the orthogonal parallelepiped EFBD:

$$\forall_{LA} = \forall_{EFBD} + \frac{1}{2}\forall_{BDAC} = \frac{l}{4}\left(l\frac{\sqrt{2}}{2}\right)w + \frac{1}{2}\left(l\frac{\sqrt{2}}{2}\right)^2 w = \frac{l^2 w}{8}\left(2+\sqrt{2}\right)$$

7. The vertical component of the hydrostatic force is equal in magnitude to the weight of the liquid directly over the surface of the gate:

$$F_y = W_{LA} = g\ \ \forall_{LA} = 0.43g\ l^2\ w$$

8. The total force on the gate is, of course, the vector sum of its components:

$$F = \sqrt{F_x^2 + F_y^2} = 0.6g\ \ w\,l^2$$

9. Now we must find the 2nd moment of inertia of the vertical projection of the gate (BC) about a horizontal axis through its centroid:

$$I_{cv} = \frac{w(BC)^3}{12} = \frac{w\,l^3\,\sqrt{2}}{48}$$

10. The depth of the center of pressure can now be calculated from:

$$y_{cp} = y_{cv} + \frac{I_{cv}}{y_{cv} A_V} = 0.67\,l$$

11. To find the line of action of the vertical component (F_y) we must find the c.g. of the volume of the water directly over the gate, since it is the weight of this water which causes the vertical force on the gate. To find the c.g. of the water directly over the gate we take the moments of the weights of the convenient shapes from step 6.

$$\sum M_{AE} = W_{LA} x_{cp} = W_1 \frac{l\frac{\sqrt{2}}{2}}{2} + W_2 (DG)\frac{\sqrt{2}}{2} = F_y x_{cp}$$

Now $\quad DG = \frac{2}{3}(DJ) = \frac{2}{3}\sqrt{(DA)^2 - (AJ)^2} = \frac{2}{3}\sqrt{\left(\frac{l\sqrt{2}}{2}\right)^2 - \left(\frac{l}{2}\right)^2} = \frac{2}{3}\left(\frac{l}{2}\right) = \frac{l}{3}$

$$W_1 = g\ \forall_{EFBD} = g\ l\frac{\sqrt{2}}{2}\left(\frac{l}{4}\right)w = 0.18\,g\ l^2 w$$

$$W_2 = g\ \forall_{ABC} = g\ \frac{1}{2}\left(l\frac{\sqrt{2}}{2}\right)^2 w = 0.25\,g\ l^2 w$$

Substituting into the moment eq.

$$(0.18g\ l^2 w)(0.35\,l) + (0.25g\ l^2 w)(0.24\,l) = (0.43g\ l^2 w)x_{cp} \Rightarrow x_{cp} = 0.29\,l$$

12. To find the reaction at A let's take the moments about a horizontal axis through B:

$$\sum M_B = 0 \Rightarrow R\left(l\frac{\sqrt{2}}{2}\right) - F_y\left(l\frac{\sqrt{2}}{2} - x_{cp}\right) - F_x\left(y_{cp} - \frac{l}{4}\right) = 0 \Rightarrow$$

$$R\,l\frac{\sqrt{2}}{2} - 0.43g\ l^2 w(0.42\,l) - 0.43g\ l^2 w(0.42\,l) = 0 \Rightarrow R = 0.51g\ l^2 w$$

13. Regarding the cylindrical gate in figure (b) we can tell just by looking at it that the reaction will be less because:

$$F_{xb} = F_{xa}$$
$$F_{yb} \prec F_{ya}$$

since both gates have the same vertical projection (BC) but the volume of the water over gate (b) is less than the volume of the water over gate (a).